The Complexity of Rivers in Triangulated Terrains

Pankaj Agarwal[1] Mark de Berg[2] Prosenjit Bose[3] Katrin Dobrint[2] Marc van Kreveld[2]
Mark Overmars[2] Marko de Groot[2] Thomas Roos[4] Jack Snoeyink[5] Sidi Yu[5]

Abstract

Triangulated surfaces are often used to represent terrains in Geographic Information Systems (GIS); one of the primary computations on terrains is determining drainage networks.

Under natural definitions of the flow of water on a terrain represented by n triangles, we show that the river network has $\Theta(n^3)$ worst-case complexity, where complexity is measured in the number of line segments that make up the network.

1 Introduction

Terrain drainage characteristics provide important information on water resources, possible flood areas, erosion and other natural processes. In natural resource management, for example, the basic management unit is the *watershed*, the area around a stream that drains into the stream. Road building, logging, or other activities carried out in a watershed all have the potential to affect the defining stream. Manual quantification of terrain drainage characteristics is a tedious and time consuming job. Fortunately, through spatial analysis of digital representations of surfaces, they can be, by and large, inferred automatically.

In this note, we survey some of the literature on computing drainage information in digital terrain models and look carefully at the definitions of river networks. Under our definition, we prove a tight cubic bound on the complexity (measured in number of line segments) of a river network. There are a number of open practical and theoretical questions.

2 Terrain Models

There are three main forms of digital terrain representations: digital contours, gridded DEM (digital elevation model storing elevations at points of a regular grid), and TINs (triangulated irregular networks). Generally, the drainage network has been computed on a gridded models of the surface [2, 6, 7, 11, 16, 18, 21, 23], often using local filters to detect potential pits, peaks and channels, much as in raster image processing. Even researchers who define drainage networks in contours [5, 24, 11] may compute them in DEMs. As with image processing, there are inherent ambiguities with the local raster approach, because the global nature of drainage on a mathematical surface is not adequately reflected.

---

[1]Duke University, [2]Utrecht University, [3]Université du Québec à Trois-Rivières, [4]ETH Zurich, and [5]University of British Columbia Departments of Computer Science.

The authors thank the Dutch Organization for Scientific Research (N.W.O.) for the PIONEER grant that supported the Utrecht Workshop on Computational Geometry and GIS, at which this work was begun. Various authors received partial support from N.W.O., the US National Science Foundation, Canada's National Science and Engineering Research Council, and the B.C. Advanced Systems Institute.

We choose to concentrate on the TIN model because it specifies an explicit mathematical surface on which fluid can flow. Even so, previous work on drainage in TINs does not always respect the physics of (non-inertial) flow on a surface: Researchers typically restrict flow to triangle edges or consider fluid to flow from triangle centroid to triangle centroid based on triangle neighbor relations [25, 22, 17, 8]. We will allow cross-triangle flows, which are important for connectivity and duality properties of the network.

The study of the drainage network on *smooth* mathematical surfaces has a long history, including mathematicians such as Cayley in 1859 [1] and Maxwell in 1870 [13]. Koenderink and van Doorn [9, 10] credit Rothe [20] with publishing the first solution in 1915; Koenderink and van Doorn's work expresses this solution in the terminology of modern differential geometry and includes examples of why other attempts at a solution (before and since) are inadequate. The fact that TINs are not smooth makes flows more combinatorial. For example, junctions can be identified as discrete critical points rather than happening in the limit.

Related to drainage networks are the so-called surface networks by Mark [12], Pfaltz [19], and Wolf [27, 28]. Also worth mentioning as related work is the computation of terrain data from hydrological information [14, 24]; hydrologists use complex models that differentiate between surface flow and subsurface flow, and therefore are also dependent on permeability of the ground, rainfall, and other quantities [22, 26].

3 Definitions for Drainage

In the Geographic Information System (GIS) literature, it is difficult to find precise definitions of the drainage network and its component rivers and ridges. In many works, the drainage network is defined only implicitly as the output of some algorithm on terrain data. Definitions that are given often depend on a certain terrain model (e.g., a grid) and cannot be translated to other terrain models. Frank et al. [4] point out that formal definitions should be used to define terrain-specific features so that properties of the structure can be established mathematically and contradictory definitions (or at least those in disagreement with other related research) can be avoided.

In this paper we use simple and general definitions that extend those of Frank et al. [4]. More details and consequences can be found in the paper by Yu et al. [29]. First, two assumptions about the downhill path traced by a drop of water:

- A1 At any point, water follows the steepest descent.
- A2 At any point, the steepest descent is unique.

Assumption A1 is standard for viscous flow; A2 is plausible on faces and is adopted by convention for edges and vertices, with the result that the path taken by a drop of water is determined by its initial point and that two paths that join never separate.

- D1 The *trickle path* of a point p on a surface S is the path that begins at p and follows the steepest descent until it reaches a local minimum or the boundary of S.
- D2 The *watershed* of a point p is the set of all points whose trickle paths contain p.
- D3 The *river network* consists of all points whose watersheds have non-zero area (more precisely, have two-dimensional Lebesgue measure).

We use two more definitions from Frank that are specific to a TIN:

- D4 A *pit* is a local minimum of the surface; a vertex with lower elevation than its incident edges and triangles.

D5 A *local channel* (or *cofluent edge*) is a triangulation edge for which both adjacent triangles contain interior points whose trickle down paths intersect the edge. See figure 1.

Under these definitions, we can begin to characterize the river network:

Lemma 3.1 *The river network in a TIN is a forest of disjoint trees whose leaves are local channels.*

Proof: The trickle path from any point on the river network is contained in the river network. Since a trickle path can only end at a pit or at the region boundary, the river network is a forest of trees.

Points on the relative interior of local channels have watersheds that than include positive area in both adjacent triangles, so they are clearly on the river network. On the other hand, a point p inside a triangle collects flow only from the points immediately above (along the steepest ascent) and a vertex p of the TIN collects flow from a set of steepest ascents in adjacent triangles and edges. In both cases, the watershed of p has zero area in the neighborhood of p; point p is on the river network only if points above are on the river. Thus, the leaves of the river network are local channels. ■

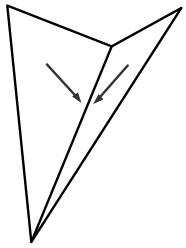

Figure 1: A local channel (cofluent edge)

4 An Upper Bound on River Network Complexity

For computation and storage of the river network, we would like to know its *complexity,* which we define as the number of line segments that are needed to represent the network. (Since trickle paths in a TIN are composed of line segments, the network can indeed be represented with segments).

Path ρ visits a sequence of triangles and edges. Within each triangle, ρ's direction is determined by the steepest descent. Thus, if ρ visits a triangle a second time, then it repeats the sequence that it visited before until it ends at the elevation of a vertex, as illustrated in figure 2.

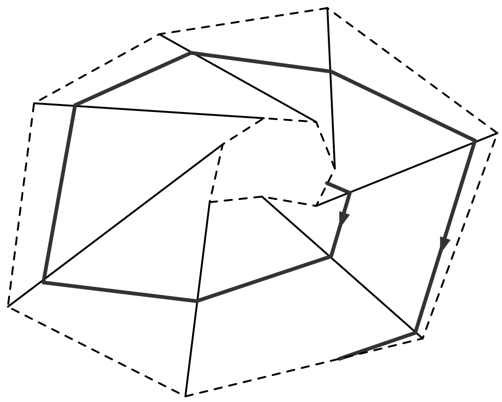

Figure 2: Repeating a triangle repeats the subsequent sequence. (This figure cannot be drawn accurately—each quadrilateral should be bounded by parallel dotted edges that are perpendicular to the path.)

By Euler's relation, we know that the number of edges of an n-vertex TIN is at most $3n$. Thus, Lemma 3.1 implies that the river network consists of a tree with $O(n)$ leaves, paths, and junctions. We must determine how many triangles (or, equivalently, triangulation edges) the river network can cross. Under our definitions, a trickle path can cross an edge of an n-vertex TIN a linear number of times. This may seem like a weak upper bound, but the next section will show that it is tight.

Lemma 4.1 *In a terrain represented by a TIN with n vertices, a trickle path meets any edge of the TIN at most $O(n)$ times.*

Proof: Break the trickle path into n subpaths by cutting at the elevation of each vertex of the TIN; let ρ denote one of these subpaths. We will show that ρ hits each edge of the TIN at most twice.

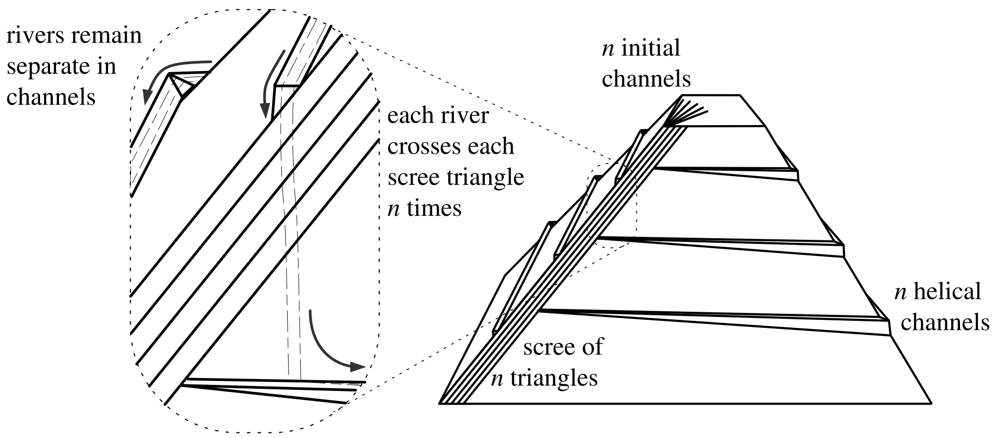

Figure 3: A construction of a TIN with $O(n)$ vertices in which n paths each have n^2 complexity. Some edges of the TIN have been omitted to increase clarity.

Consider now the edge in the repeating sequence that has the shallowest slope, where the *slope* of a segment in 3-d is the ratio of the positive difference in z coordinates with the length of the projection in the xy plane. If ρ visits this edge first at point p and lower at point q, then we can derive a contradiction: The segment $\overline{pq}$ is shorter than the subpath of ρ from p down to q. This subpath, however is steeper than $\overline{pq}$—because ρ follows the steepest descent, each triangle on which the subpath flows must be steeper than the preceding edge, and all must be steeper than $\overline{pq}$ on the shallowest edge. A longer path from p having a steeper slope than $\overline{pq}$, however, cannot end at q.

Since the shallowest edge cannot be visited twice, no other edge can be visited more than twice. ■

5 A Worst-case Example

Lemma 4.1 implies that the worst-case complexity of all rivers in a TIN with n vertices is $O(n^3)$. This bound can actually be achieved by an ancient Egyptian construction technique shown in figure 3.

We describe this construction qualitatively. Start with the frustrum of a pyramid. Along one edge of the pyramid, form a *scree* (a term meaning "rocky slope") of n long triangles. On the top, carve n local channels that will catch water and start n rivers near the scree. Each river will cross the scree n times.

Form n gently-sloping helical channels that meet the rivers after the scree, carry them around the pyramid and back to the scree. Each channel begins with a nearly-flat top whose steepest descent angles slightly away from the pyramid face, so that rivers that join the channel separately remain separate. At a turn, the two top triangles of the channel share a vertex at the outside corner and a small, near-vertical triangle interpolates between them, forming separate waterfalls for separate rivers.

After triangulating between the channels and the scree, the resulting terrain has $O(n)$ vertices and the n initial rivers each cross the n scree triangles n times. We summarize in the following theorem.

Theorem 5.1 *In a TIN with n vertices, the worst-case complexity of a river is $\Theta(n^2)$ and of all rivers is $\Theta(n^3)$.*

6 Directions for Future Research

While a $\Theta(n^3)$ worst-case bound may sound discouraging for algorithm development, it is clear that our pyramid example is unlikely to arise in real data. In conjunction with Facet Decision Systems of Vancouver, we are addressing the question of the empirical complexity of the river networks in terrain data provided by the B.C. Ministry of Environment. Even if complex networks do arise, data structuring ideas such as persistent structures for similar lists [3] or Mount's "bundling" [15] can reduce the storage demands.

Interesting from a theoretical viewpoint is the question of whether special triangulations can guarantee better river complexities. Is it possible to prove an $O(n^2)$ bound for the complexity of a river network in a Delaunay triangulation or in some other triangulation with well-shaped triangles?

Our drainage model can be extended to take into account the absorption of water by different ground types. It can also be extended to take into account the fact that water has volume and can collect—one clearly should do so in terrains because errors in data or in sampling can create spurious local minima that are shallow pits. Any significant volume of water would collect to fill these and then spill over the lowest pass to rejoin the main network. Also, rivers whose flow is less than a certain volume can be pruned from the network. Can one compute the rivers that remain in time proportional to their complexity—i.e., can one compute the main rivers without inspecting all the tributaries?

References

[1] A. Cayley. On contour and slope lines. *Lond. Edin. Dublin Phil. Mag. and J. of Sci.*, 18(120):264–268, 1859.

[2] D. H. Douglas. Experiments to locate ridges and channels to create a new type of digital elevation model. *The Canadian Surveyer*, 41(3):373–406, Autumn 1986.

[3] J. R. Driscoll, N. Sarnak, D. D. Sleator, and R. E. Tarjan. Making data structures persistent. *J. Comp. Sys. Sci.*, 38:86–124, 1989.

[4] A. U. Frank, B. Palmer, and V. B. Robinson. Formal methods for the accurate definition of some fundamental terms in physical geography. In *Proc. 2nd Intl. Symp. Spatial Data Handling*, pages 585–599, 1986.

[5] M. F. Hutchinson. Calculation of hydrologically sound digital elevation models. In *Proc. 3rd Intl. Symp. Spatial Data Handling*, pages 117–133, 1988.

[6] S. K. Jenson. Automated derivation of hydrologic basin characteristics from digital elevation model data. In *Proc. AUTO-CARTO 7*, pages 301–310. ASP/ACSM, 1985.

[7] S. K. Jenson and J. O. Domingue. Extracting topographic structure from digital elevation data for geographic information system analysis. *Photogrammetric Engineering and Remote Sensing*, 54(11):1593–1600, Nov. 1988.

[8] N. L. Jones, S. G. Wright, and D. R. Maidment. Watershed delineation with triangle-based terrain models. *Journal of Hydraulic Engineering*, 116(10):1232–1251, Oct. 1990.

[9] J. J. Koenderink and A. J. van Doorn. Local features of smooth shapes: Ridges and courses. In *Geometric Methods in Computer Vision II*, volume 2031, pages 2–13. SPIE, 1993.

[10] J. J. Koenderink and A. J. van Doorn. Two-plus-one-dimensional differential geometry. *Pat. Recog. Letters*, 15:439–443, May 1994.

[11] I. S. Kweon and T. Kanade. Extracting topological terrain features from elevation maps. *Comp. Vis. Graph. Image Proc.*, 59(2):171–182, Mar. 1994.

[12] D. M. Mark. Topological properties of geographic surfaces: Applications in computer cartography. In G. Dutton, editor, *First Int'l Adv. Study Symp. on Topological Data Structures for Geo. Info. Sys.*, volume 5. Harvard, 1978.

[13] J. C. Maxwell. On hills and dales. *Lond. Edin. Dublin Phil. Mag. and J. of Sci.*, 40(269):421–425, 1870.

[14] D. G. Morris and R. W. Flavin. A digital terrain model for hydrology. In *Proc. 4th Intl. Symp. Spatial Data Handling*, pages 250–262, 1990.

[15] D. M. Mount. Storing the subdivision of a polyhedral surface. In *Proceedings of the Second Annual Symposium on Computational Geometry*, pages 150–158, Yorktown Heights, New York, June 1986. ACM SIGGACT.

[16] J. F. O'Callaghan and D. M. Mark. The extraction of drainage networks from digital elevation data. *Comp. Vis. Graph. Image Proc.*, 28:323–344, 1984.

[17] O. L. Palacios-Velez and B. Cuevas-Renaud. Automated river-course, ridge and basin delineation from digital elevation data. *Journal of Hydrology*, 86:299–314, 1986.

[18] T. K. Peucker and N. Chrisman. Cartographic data structures. *Amer. Cartog.*, 2(1):55–69, 1975.

[19] J. L. Pfaltz. Surface networks. *Geog. Anal.*, 8:77–93, 1976.

[20] R. Rothe. Zum problem des talwegs. *Sitz. ber. d. Berliner Math. Gesellschaft*, 14:51–69, 1915.

[21] W. W. Seemuller. The extraction of ordered vector drainage networks from elevation data. *Comp. Vis. Graph. Image Proc.*, 47:45–58, 1989.

[22] A. T. Silfer, G. J. Kinn, and J. M. Hassett. A geographic information system utilizing the triangulated irregular network as a basis for hydrologic modeling. In *Proc. Auto-Carto 8*, pages 129–136, 1987.

[23] S. Takahashi, T. Ikeda, Y. Shinagawa, T. L. Kunii, and M. Ueda. Algorithms for extracting correct critical points and constructing topological graphs from discrete geographical elevation data. In F. Post and M. Göbel, editors, *Eurographics '95*, volume 14, pages C–181–C–192. Blackwell Publishers, 1995.

[24] L. Tang. Automatic extraction of specific geomorphological elements from contours. In *Proc. 5th Intl. Symp. Spatial Data Handling*, pages 554–566, 1992.

[25] D. M. Theobald and M. F. Goodchild. Artifacts of tin-based surface flow modeling. In *Proc. GIS/LIS'90*, pages 955–964, 1990.

[26] A. K. Turner. The role of 3-d gis in subsurface characterization for hydrological applications. In J. F. Raper, editor, *Three Dimensional Applications in GIS*, pages 115–127. Taylor & Francis, Bristol, 1989.

[27] G. W. Wolf. Metric surface networks. In *Proc. 5th Intl. Symp. Spatial Data Handling*, pages 844–856, 1990.

[28] G. W. Wolf. Hydrologic applications of weighted surface networks. In *Proc. 5th Intl. Symp. Spatial Data Handling*, pages 567–579. IGU Commision on GIS, 1992.

[29] S. Yu, M. van Kreveld, and J. Snoeyink. Drainage queries in TINs: From local to global and back again. Accepted to the 7th Intl. Symp. Spatial Data Handling, 1996.