Feasibility of Design in Stereolithography

B. Asberg[1], G. Blanco[2], P. Bose[3], J. Garcia-Lopez[2], M. Overmars[1], G. Toussaint[3], G. Wilfong[4], and B. Zhu[3]

Utrecht University, Utrecht, Netherlands,
Escuela Universitaria de Informatica Madrid, Spain,
McGill University, Montreal, Canada,
AT & T Bell Labs, Murray Hill, New Jersey, U.S.A.

Abstract. We study the feasibility of design for a layer-deposition manufacturing process called *stereolithography* which works by controlling a vertical laser beam which when targeted on a photocurable liquid causes the liquid to harden. Given an object (modeled as a polygon or a polyhedron), we give algorithms that decide in $O(n)$ time whether or not the object can be constructed using stereolithography. Furthermore, if the answer is in the affirmative, the algorithm reports a description of all the positions in space in which the object can be made. We also determine feasibility of both polygonal and polyhedral objects constructed using variable-angle stereolithography. We give an $O(n)$ time algorithm for polygons and $O(n \log n)$ as well as $O(n)$ time algorithms for polyhedra. We show that objects formed using variable-angle stereolithography can also be constructed using another manufacturing process known as *injection molding*. Finally, we show that the polyhedral objects formed by stereolithography are closely related to polyhedral terrains which are important structures in geographic information systems (GIS) and computational geometry. In fact, our algorithms recognize whether a polyhedral surface is a terrain that allows overhangs, thus initiating the study of more realistic terrains than the standard ones considered in geographic information systems.

1 Introduction

A CAD/CAM system allows an engineer to design a product using the CAD system, store a description of the product in a data base and from the description in the data base, derive a sequence of actions for the computer controlled machines (CCMs) in order to manufacture the product. The class of products that can be realized by a particular system is clearly constrained by the versatility of the CAD system and the number of each type of CCM available. Therefore a useful feature of such a system would be the ability to decide feasibility of manufacture, that is, it would be useful to be able to decide whether or not a particular design is realizable by an existing configuration of CCMs.

We consider the problem of deciding whether or not a design is feasible for a CAD/CAM system developed and patented by 3D Systems of Sylmar, CA that employs a process called *stereolithography*. (See Figure 1). The components of the stereolithography manufacturing process consist of a vat of liquid photocurable plastic, a computer controlled table T on a stand S that can be moved up and down in the vat and a laser L above the vat that can shine on the surface of the liquid plastic and can move in a horizontal plane. The system works as follows. At the first step the table is just below the surface of the plastic and the laser is controlled to move about so that the light shines on the surface of the plastic and *draws* the bottom-most cross-section of the object A being built. When the laser light contacts the plastic, the plastic solidifies and so the first cross-section of the object is formed and rests on the table. At the next step the table is lowered a small amount to allow liquid to cover the hardened layer and the laser then draws the next cross-section of the object. The light from the laser penetrates the liquid just deep enough so that this cross-section is welded to the lower cross-section produced at the previous step. This process is repreated until the entire object is formed. The direction given by a normal to the table pointing to the laser is called the *direction of formation* for the object.

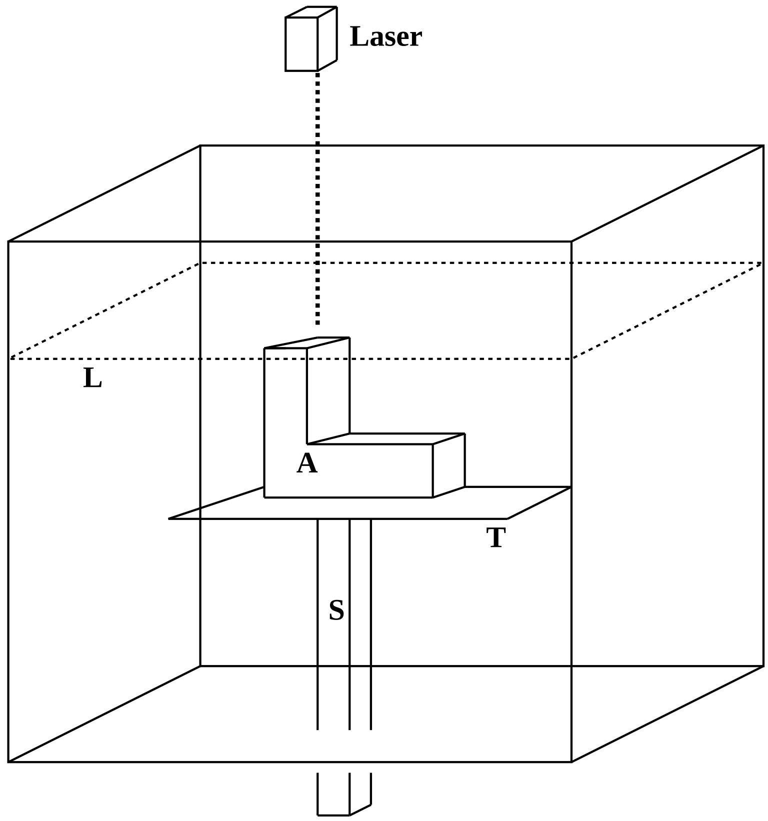

Fig. 1.

There are some objects that can be formed only if the direction of formation is chosen correctly. For example, in Figure 2, the object (a) can not be formed in the position shown. Consider what occurs when the cross-section is reached where the surface S lies. The surface S is not supported below and so as it is formed it sinks to the level of the table. However, if the object is formed in the opposite direction as in Figure 2 object (b) then stereolithography will succeed. Naturally, there are some objects that can not be formed using stereolithography regardless of the direction of formation chosen.

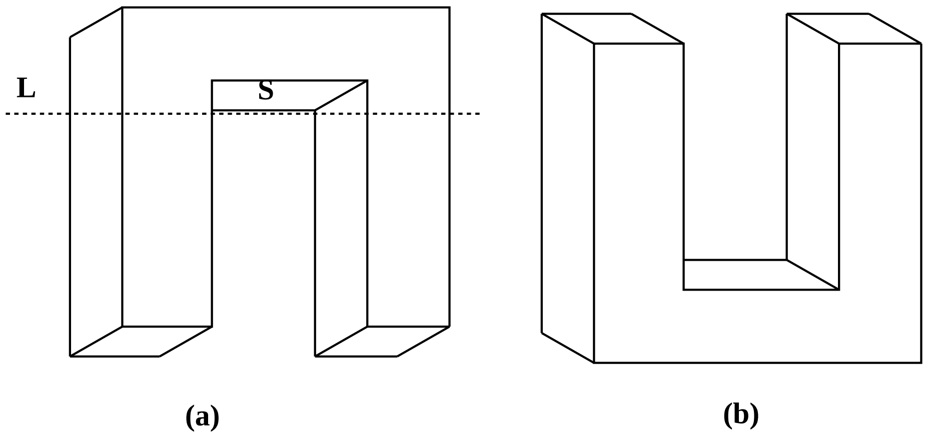

Fig. 2.

We study the problem of deciding whether or not an object can be formed using stereolithography. That is, we decide if a direction of formation exists that will result in the successful construction of the object. Such a direction will be called a *valid direction of formation*. The question is studied for objects modeled as polygons and polyhedra. We provide an $O(n)$ time algorithm for finding a valid direction of formation where n is the number of vertices of the object. Furthermore, if the object is feasible, we report a description of all the positions in space in which the object can be made. If we allow the laser beam to slightly deviate from vertical in the stereolithography process (referred to as *variable-angle stereolithography*), we determine feasibility of polygonal objects in $O(n)$ time. For polyhedral objects, we present a simple $O(n \log n)$ time algorithm as well as an optimal $O(n)$ time algorithm to determine whether or not the object may be formed using variable-angle stereolithography. We show that objects formed using variable-angle stereolithography can also be constructed using injection molding ([1], [2], [3], [4], [9]). Finally we show that the polyhedral objects formed by stereolithography are closely related to polyhedral terrains which are important structures in geographic information systems ([14], [15]) and computational geometry ([5], [7], [8]). In fact, our algorithms recognize whether a polyhedral surface is a terrain that allows overhangs, thus initiating the study of more realistic terrains than the standard ones considered in geographic information systems.

2 Notation and Preliminaries

The objects studied are simple polygons and simple polyhedra as defined in [12]. The object will always be oriented so that the direction of formation being discussed is the positive y-direction. Therefore a direction of formation defines the orientation of the object. A polygonal object is assumed to rest on the x-axis and a polyhedral object is assumed to lie on the plane defined by $y = 0$. For a given object A and direction of formation d, let A_d denote the object oriented and positioned according to d. For $y_0 \geq 0$, let $A_d(y_0)$ be the intersection of A_d with the line $y = y_0$ for polygonal objects and the plane $y = y_0$ for polyhedral objects. We refer to $A_d(0)$ as the *base* of the object (with respect to d). A point p of the object with y-coordinate y_0 is said to be *supported* (with respect to a particular direction of formation) if all the points with x (and z) coordinates the same as p and positive y coordinate less than y_0 are in the object. The cross-sections of the object are assumed to be infinitesimally thin and so direction d is a valid direction of formation for an object if the resulting orientation of the object is such that all points in the object are supported.

Lemma 1. *If d is a valid direction of formation for polygon (polyhedron) A, then $A_d(0)$ is some edge (face) of A.*

Let $d(F)$ denote the direction normal to edge (face) F pointing into the polygon (polyhedron). The above lemma implies that these are the only directions that need to be verified. An edge (face) F of polygon (polyhedron) A is said to be a *valid base* if $d(F)$ is a valid direction of formation. A point p in $A_{d(F)}$ is said to be *vertically visible* from F if the vertical line segment from p to F is contained in $A_{d(F)}$. Thus, we have the following lemma.

Lemma 2. *A edge (face) F of $A_{d(F)}$ is a valid base if and only if all points in $A_{d(F)}$ are vertically visible from F.*

An open set of points H is called a hole in object A if $H \subset A$ and $H \cap A = 0$. We note the following property.

Lemma 3. *If A is feasible then A contains no holes.*

3 Polygonal objects

In this section we consider the two-dimensional problem where the object A we wish to form is a polygon. Let $e_1, e_2, \ldots, e_n$ be a clockwise ordering of the edges around A and for $1 \leq i \leq n$, let θ_i be the angle formed by e_i and e_{i+1} in the interior of A (where $e_{n+1} = e_1$). Note that θ_0 will mean θ_n. If edge e_i is such that θ_{i-1} and θ_i are both less than or equal to $\pi/2$ then e_i is called an *acute edge*. If e_i is an acute edge and at least one of θ_{i-1} or θ_i is strictly less than $\pi/2$ then e_i is said to be a *strictly acute edge*.

Lemma 4. *If e_i is a valid base then e_i is acute.*

The following lemma shows that for a convex object A there is a simple linear time test to find a valid base for A or report that none exists.

Lemma 5. *Given a convex polygon A, the edge e_i is a valid base if and only if e_i is acute.*

The characterization of convex objects in lemma 5 implies that a simple examination of the angles between the edges of a convex object is sufficient to find a valid base if one exists or report that the object is not feasible. For a non-convex object however such local tests on the angles will be insufficient to determine the feasibility of an object. However the angles will be used to narrow the search for a valid base.

Lemma 6. *If polygon A is feasible and edge e_i of A is strictly acute then e_i, e_{i-1} or e_{i+1} is a valid base.*

Lemma 4 guarantees that an acute edge e_i exists if A is feasible and Lemma 6 says that if a strictly acute edge e_i exists then it is sufficient to test e_i, e_{i-1} and e_{i+1} for a valid base. We now consider what happens when both θ_{i-1} and θ_i are equal to $\pi/2$. If A_{e_i} contains a horizontal edge whose points have y-coordinates at least as great as any point in A_{e_i} and such that the x-coordinates of all points in A_{e_i} lie between the x-coordinates of the endpoints of the edge then we label the edge $e_{top}(i)$.

Lemma 7. *If A is feasible and e_i is an acute edge such that $\theta_{i-1} = \theta_i = \pi/2$ then at least one of e_i, e_{i-1}, e_{i+1} or $e_{top}(i)$ (if it exists) is a valid base.*

Theorem 8. *Polygon A is feasible if and only if A contains an acute edge e_i and at least one of e_{i-1}, e_{i-1}, e_{i+1} or $e_{top}(i)$ (if it exists and $\theta_{i-1} = \theta_i = \pi/2$) is a valid base.*

Determining whether or not a polygon has an acute edge can be easily achieved in $O(n)$ time, where n is the number of vertices in the polygon. Thus, in $O(n)$ time, the number of possible valid base can be reduced to 3 or 4.

Lemma 9. *Deciding if an edge e_i of a polygon A is a valid base can be done in $O(n)$ time.*

Therefore, we can test a polygon A for feasibility and find a valid base in $O(n)$ time where n is the number of vertices of A.

Theorem 10. *In $O(n)$ time the feasibility of a polygonal object with n vertices can be determined and all valid bases identified when the object is feasible.*

4 Polyhedral objects

In this section we consider the three-dimensional case where the object is a simple polyhedron. We want to find a face f of polyhedron A that is a valid base or determine that A is not feasible.

The following notation will be used in this section. Let A be a polyhedron with n vertices. For face f of A, let $f(1), f(2), \ldots, f(k_f)$ be the faces of A that share at least one edge with f. Let $\theta_i(f)$ be the angle interior to A between the plane P containing f and the plane $P(i)$ containing $f(i)$ about the line of intersection of P and $P(i)$. If $\theta_i(f) \leq \pi/2$ for all i, $1 \leq i \leq k_f$, then f is called an *acute face*. If f is acute and for some i, $\theta_i(f) < \pi/2$, then f is said to be a *strictly acute face*. We show several properties analogous to those in the previous section that will give rise to a linear time feasibility testing algorithm.

Lemma 11. *If face f is a valid base for polyhedron A then f is acute.*

In the special case of convex polyhedra, we see that a simple local test on each face suffices to determine feasibility.

Lemma 12. *Let A be a convex polyhedron. Face f is a valid base if and only if f is acute.*

We now turn our attention to polyhedral objects that are not necessarily convex. We show that if the polyhedral object A has a strictly acute face f, then f or one of its adjacent faces must be a valid base if the object is feasible.

Lemma 13. *If polyhedron A is feasible and f is a strictly acute face then $f, f(1), \ldots,$ or $f(k_f)$ is a valid base.*

If f is not strictly acute, we define f_{top} analogously to $e_{top}(i)$ in the previous section. We have the following lemma.

Lemma 14. *If polyhedron A is feasible and f is an acute, but not strictly acute, face then $f, f(1), \ldots, f(k_f)$ or f_{top} is a valid base.*

These results were sufficient in the two-dimensional case to reduce the number of candidate bases to at most 4. Unfortunately, in the 3-dimensional case, an acute face f may have $O(n)$ adjacent faces. Nevertheless, we have the following theorem.

Theorem 15. *Polyhedron A is feasible if and only if A contains an acute face f and at least one of the following is a valid base of A: f, f_{top} (if it exists) and at most 4 faces adjacent to f.*

Thus, in $O(n)$ time, where n is the number of vertices in polyhedron A, the number of possible valid bases can be reduced to at most 6.

Lemma 16. *Deciding if a face f of polyhedron A is a valid base can be done in $O(n)$ time, where n is the number of vertices of A.*

Therefore, we can test a polyhedron A for feasibility and find a valid base in $O(n)$ time where n is the number of vertices of A.

Theorem 17. *In $O(n)$ time the feasibility of a polyhedral object with n vertices can be determined and a valid base identified when the object is feasible.*

5 Variable-Angle Stereolithography

In this section, we characterize the objects, both polygonal and polyhedral, that can be formed with variable-angle stereolithography. Intuitively, in variable-angle stereolithography, one can imagine that as each layer is glued on by the laser, the topmost layer cannot hang over the previous layer by more than the freedom allotted by some constant angle ω. More formally, we say that a point p with y-coordinate y_0 is supported with respect to the direction of formation if there exists a point q with positive y coordinate less than y_0 such that the line segment $\overline{qp}$ is contained in object and the smaller angle between the direction of formation and the vector $\mathbf{qp}$ is less than or equal to ω. Clearly, ω must be less than $\pi/2$. Notice that variable-angle stereolithography is a generalization of stereolithography. The two are equivalent when ω is zero.

5.1 Polygonal Objects

Although the parameter ω enlarges the class of objects that can be formed, we have the following two key theorems that allow us to efficiently recognize the polygons that can be formed using variable-angle stereolithography.

Theorem 18. *For fixed ω, a convex polygon has a constant number of edges that can be valid bases. These edges can be obtained in $O(n)$ time.*

Theorem 19. *If a polygon A is feasible, then every valid base of A is also a valid base of the convex hull of A.*

The convex hull of a simple polygon can be computed in $O(n)$ time ([16], [10]). Thus we have the following theorem.

Theorem 20. *The feasibility of a simple polygon in variable-angle stereolithography can be determined in $O(n)$ time.*

5.2 Polyhedral Objects

With respect to polyhedral objects, the parameter ω not only enlarges the class of objects that can be formed but also renders the determination of feasibility more difficult than the two-dimensional case. We have the following two key theorems that are analogous to the two-dimensional case.

Theorem 21. *For fixed ω, a convex polyhedron has a constant number of faces that can be valid bases. These faces can be obtained in $O(n)$ time.*

Theorem 22. *If a polyhedron A is feasible, then every valid base of A is also a valid base of the convex hull of A.*

The complexity of computing the convex hull of a polyhedron is $O(n \log n)$ ([12]), thus the algorithm derived by computing the convex hull is sub-optimal. However, we are able to circumvent the computation of the convex hull and obtain the following theorem.

Theorem 23. *The feasibility of a simple polyhedron can be determined in $O(n/\tau^2)$, where $\tau = \pi/2 - \omega$.*

6 Relation to Injection Molding

A well-known technique used in the manufacturing of goods is injection molding. A mold, as defined in [6], refers to the whole assembly of parts that make up a cavity into which liquid is poured to give the shape of the desired component when the liquid hardens. Given a mold (modeled as a polyhedron), establishing whether there exists an orientation that allows the filling of the mold using only one pin gate (the pin gate is the point from which the liquid is injected into the mold) as well as determining an orientation that allows the most complete fill are two major problems in the field of injection molding. These and other related problems in molding have been studied by Bose and Toussaint [1], [2], Bose, Van Kreveld and Toussaint [3], [4], Fekete and Mitchell [9], and Rosenbloom and Rappaport [17] from an algorithmic and geometric point of view.

In [4], it is shown that given an arbitrary polyhedron on n vertices, in $O(n^2)$ time all the orientations that allow the polyhedron to be filled from one pin gate can be found. It is also shown in [4] that this problem is n^2-hard [13], which leads one to believe that improving this time complexity will be difficult. However, for the restricted case of a polyhedron that can be built using variable-angle stereolithography, we have the following theorem relating the two manufacturing processes.

Theorem 24. *A polyhedral object formed by variable-angle stereolithography can be formed by injection molding using only one pin gate.*

7 Relation to Polyhedral Terrains

Polyhedral terrains are important structures in geographic information systems ([15], [14]) and computational geometry ([7], [8], [5]). A polyhedral terrain is defined as follows.

A terrain T is a two-dimensional triangulated polyhedral surface with n vertices $V = \{v_1, v_2, \ldots, v_n\}$. Each vertex v_i is specified by three real numbers (x_i, y_i, z_i) which are its cartesian coordinates and z_i is referred to as the height of vertex v_i. It is convenient to assume that z_i is non-negative so that if the

X-Y plane is associated with *sea-level*, no points on the terrain are below sea-level. Let $P = \{p_1, p_2, \ldots, p_n\}$ denote the orthogonal projections of the points $V = \{v_1, v_2, \ldots, v_n\}$ on the X-Y plane, i.e., each point p_i is specified by the two real numbers (x_i, y_i). It is assumed that the set $P = \{p_1, p_2, \ldots, p_n\}$ is in general position, i.e., no three points are collinear and no four are co-circular so that the projections of the edges of the polyhedral surface onto the X-Y plane determine a triangulation of P (hence the term triangulated polyhedral surface). We refer to the triangulation as the underlying triangulated planar graph associated with the terrain. Therefore we can view a terrain T as the graph of a polyhedral function $z = F(x, y)$, defined over the convex hull of P. Sometimes a polyhedral terrain is assumed to be a monotone polyhedral surface, i.e., a polyhedral surface having exactly one intersection with every vertical line [7]. In our case we assume the stronger condition that the intersection of every vertical line in the interior of convex hull of P with the polyhedral surface is a single point. Intuitively, a monotone terrain admits vertical walls whereas our definition does not. Since the orthogonal projection of T onto the X-Y plane is a planar straight-line subdivision or map, it follows that T has $O(n)$ edges and $O(n)$ triangular faces. A polyhedron P is a solid terrain provided that it can be positioned such that it is vertically visible from one of its faces.

We have the following theorem.

Theorem 25. *Determining if a polyhedron P is a solid terrain can be achieved in $O(n)$ time.*

In fact, the polyhedra that can be formed by variable-angle stereolithography provide a generalization of the standard polyhedral terrains studied in geographic information systems. Our algorithms recognize whether a polyhedral surface is a terrain that allows overhangs, a more realistic terrain.

8 Conclusions

We have presented linear time algorithms for determining whether or not an object (modeled as a polygon or polyhedron) can be manufactured using stereolithography. For feasible objects our algorithms report a description of all the positions in space in which the object can be made. We also determine feasibility of both polygonal and polyhedral objects constructed using variable-angle stereolithography. We give an $O(n)$ algorithm for polygons and an $O(n \log n)$ as well as an $O(n)$ algorithm for polyhedra. We show that objects formed using variable-angle stereolithography can also be constructed using another manufacturing process known as *injection molding*. Finally, we show that the polyhedral objects formed by stereolithography are closely related to polyhedral terrains, and introduce a new more general and realistic class of polyhedral terrains.

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