On the Sectional Area of Convex Polytopes*

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1 Introduction

A function $f : \mathcal{R} \to \mathcal{R}$ is *unimodal* if it increases to a maximum value and then decreases. It is *strictly unimodal* if the increase and decrease are strict. To be precise, f is strictly unimodal iff for all reals $x < y$, the value $v = \min\{f(x), f(y)\}$ is either the global minimum or maximum of f or $v < f(z)$ for all $z \in (x, y)$.

Unimodality is important for the design of efficient search algorithms because it permits *prune-and-search* strategies. E.g., Chazelle and Dobkin [2] showed that the perpendicular distance from a line ℓ to an n-vertex convex polygon Q is bimodal. From this, unimodal functions can be constructed and the farthest point from ℓ can be computed in $O(\log n)$ time. Unimodality can also simplify proofs.

It was shown in 1973 that if $A(x)$ is the length of the intersection of a convex polygon Q with the vertical line through x, then $A(x)$ is a strictly unimodal function [4]. We note that this generalizes to higher dimensions: the area in $\mathcal{R}^3$ (or volume in $\mathcal{R}^d$) of the intersection of a convex polytope K and a hyperplane $h(x) = \{(x, x_2, \ldots, x_d) \mid \forall x_i \in \mathcal{R}\}$ is a strictly unimodal function $A(x)$. (If the plane is defined by rotation instead of translation, then there are convex polytopes for which sectional area is not unimodal.) Prune-and-search can be used to compute the intersection with maximum area (volume) in time proportional to the size of K, if K is stored with its complete facial lattice.

For $\mathcal{R}^3$ our algorithm has an application to shape matching: Given convex polygons P and Q and a direction in which to translate P, one can find the translation having maximum overlap with Q in linear time.

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2 Unimodality of Sectional Area

Let K be a convex polyhedron in $\mathcal{R}^d$. Let $h(x)$ be the hyperplane consisting of all points with first coordinate equal to x. Let $A(x)$ denote the area (or volume, if $d > 3$) of the intersction $K \cap h(x)$. We show that $A(x)$ is a strictly unimodal function; it need not be convex or concave. This has also been observed by B. Chazelle and D. Dobkin (private communication).

The proof of strict unimodality rests on the Brunn-Minkowski Theorem, a powerful inequality for *mixed volumes* in $\mathcal{R}^{d-1}$ [6]. In this theorem, polytopes are considered as sets of vectors and linear combinations of polytopes give new polytopes by scalar multiplication and Minkowski sums of sets of vectors. Let $A(Q)$ denote the area (or volume) of a polytope Q.

Theorem 2.1 (Brunn-Minkowski) *Suppose that Q_1 and Q_2 are convex polytopes in $\mathcal{R}^{d-1}$. Then $\sqrt[d-1]{A(\lambda Q_1 + (1-\lambda)Q_2)} \geq \lambda \sqrt[d-1]{A(Q_1)} + (1-\lambda) \sqrt[d-1]{A(Q_2)}$, for all $0 < \lambda < 1$. Equality holds if and only if Q_1 and Q_2 are homothetic—that is, if $Q_2 = \lambda Q_1 + p$ for some $\lambda > 0$ and $p \in \mathcal{R}^{d-1}$.*

An easy corollary establishes strict unimodality.

Corollary 2.2 *The sectional area $A(x)$ is a strictly unimodal function.*

We note that $A(x)$ need not be either convex or concave. Also, if instead of translating a plane across a convex polyhedron, we rotate a plane about some axis, then sectional area is no longer unimodal. In fact the sectional area can have $\Omega(n)$ local maxima and minima.

3 Algorithm for Sectional Area

Given a convex polytope $K \subset \mathcal{R}^d$ that is described by its facial lattice, we can compute the maximum sectional volume $A(x)$ in linear time: We represent K as a sum of positive and negative simplices, each with sectional volume functions that are piecewise polynomials of degree d. Unimodality allows us to use prune-and-search to narrow the interval for the maximum section and replace simplices that have no vertices in the interval by their volume functions.

Shermer [7] gave a similar algorithm for bisecting a polygon P in linear time, which relied on a trapezoidation of P. We observe that the trapezoids (or simplices) can be both positive and negative; they need not be a partition of P. This allows for easier generalization of his algorithm to higher dimensions. We also replace his two median searches per step by one.

To begin, we need the formula for the sectional volume of a simplex.

Lemma 3.1 ([5]) *In $\mathcal{R}^d$, the sectional volume of a simplex is a degree d polynomial between planes containing vertices.*

Theorem 3.2 *The maximum sectional volume $A(x)$ of a convex polytope $K \subset \mathcal{R}^d$ can be computed in linear time, if the facial lattice of K is given.*

Proof: Form a linear-size sum of simplices $K = \sum_i \pm S_i$ by bottom vertex triangulation or any other way of connecting facets to an origin.

Let interval $[x_1, x_2]$ contain the x coordinates of vertices of K. Let $\mathcal{S} \subset \{S_i\}$ denote the set of simplices with vertices having x coordinates in the open interval (x_1, x_2). Initialize a degree-d polynomial, $A_D(x) = 0$, to record the volume function for simplices of $\overline{\mathcal{S}} = \{S_i\} \setminus \mathcal{S}$.

List the x-coordinates of vertices in $[x_1, x_2]$ with multiplicity equal to the number of times the vertex appears in $\mathcal{S}$. Find the median x coordinate, $\tilde{x}$, and evaluate the derivative $A'(x)$ by evaluating $A'_D(x)$ and $\sum_{S_i \in \mathcal{S}} A'(S_i)$.

Next, assign $x_1 = \tilde{x}$ or $x_2 = \tilde{x}$ so that $[x_1, x_2]$ still contains the maximum. Some simplices may no longer have vertices in the open interval (x_1, x_2); we remove them from $\mathcal{S}$ and add their volume functions to polynomial A_D. If $\mathcal{S}$ becomes empty, then we maximize $A_D(x)$ over $x \in [x_1, x_2]$, otherwise we continue with the previous paragraph.

The amount of work in each step is proportional to the number of vertices of $\mathcal{S}$ in $[x_1, x_2]$, since each simplex has constant complexity. At each step, this number is halved. Thus, the total is a geometric series that sums to linear. ∎

With a different polynomial, this algorithm computes the plane $h(x)$ bisecting a (non-convex) polyhedron.

4 Shape Matching

A corollary of this result has consequences in shape matching in the plane. Given two convex polygons, P and Q, and a direction v, we can compute the translation of P in the direction v that has maximum intersection with Q or, equivalently, that minimizes the symmetric difference of $P + \alpha v$ and Q. De Berg et al. [3] have recently developed an $O(n \log n)$ algorithm to find the translation of P in any direction that has the maximum overlap with Q; their algorithm depends on this subroutine.

Corollary 4.1 *Given convex polygons P and Q, the magnitude α of the translation along a given vector v that maximizes the area of intersection $(P + \alpha v) \cap Q$ can be computed in linear time.*

Proof: Add an α coordinate perpendicular to the xy-plane. Extend Q to a right cylinder in the direction $(0, 0, 1)$ and P to a slanted cylinder in the direction $(v.x, v.y, 1)$. The intersection of these two cylinders is a polyhedron K.

Each vertex of K comes from a line through a vertex of P or Q that is parallel to its cylinder axis an that intersects a plane of the other cylinder. Because the cylinders are convex, each line gives rise to at most two vertices; the size of K is proportional to the sizes of P and Q. Chazelle's algorithm [1] says that we can compute K in linear time, but we can also do so with less machinery: Decompose P and Q into trapezoids in direction v and merge trapezoidations to compute K in linear time. A decomposition of K into positive and negative simplicies can also be easily obtained from the trapezoids that now make up its boundary.

The maximum section of K that is parallel to the xy plane has α coordinate of the desired magnitude. ∎

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