On Flips in Triangulations

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Resumen

We review a selection of results concerning edge flips in triangulations concentrating mainly on various aspects of the following problem: Given two different triangulations, how many edge flips are necessary and sufficient to transform one triangulation into another. We study the problem both from a combinatorial perspective (where only a combinatorial embedding of the triangulation is specified) and a geometric perspective (where the triangulation is embedded in the plane, vertices are points and edges are straight-line segments). We highlight both the similarities and differences of the two settings as well as outline some open problems.

1 Introduction

An *edge flip* in a graph is the operation of removing one edge and inserting a different edge such that the resulting graph remains in the same graph class. The class of graphs we are interested in are triangulations[1]. This simple operation generates many interesting questions about triangulations. For example, what is the maximum number of edges that can be flipped in any triangulation? Is the class of triangulations closed under the flip operation, i.e. given a triangulation T_1 and a different triangulation T_2 of equal size, does there always exist a finite sequence of edge flips that transforms T_1 into a triangulation isomorphic to T_2? Wagner [14] answered this question in the affirmative. This affirmative answer led to other intriguing questions such as given two triangulations, what is the shortest sequence of edge flips that transforms one triangulation into the other? How quickly can such a sequence be computed? What is the pair of triangulations that requires the longest sequence of edge flips, i.e. what is the diameter of the flip graph? If edges can be flipped simultaneously, are there shorter sequences? What is the maximum number of edges that can be flipped simultaneously in a triangulation? All of these questions and many other variants have been addressed in the literature. In Section 2, we present a brief review of some of the main results in this area followed by a discussion of some open issues that still need to be addressed.

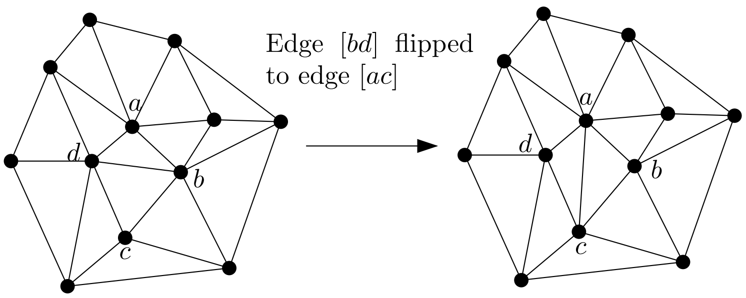

Figure 1: Example of an edge flip.

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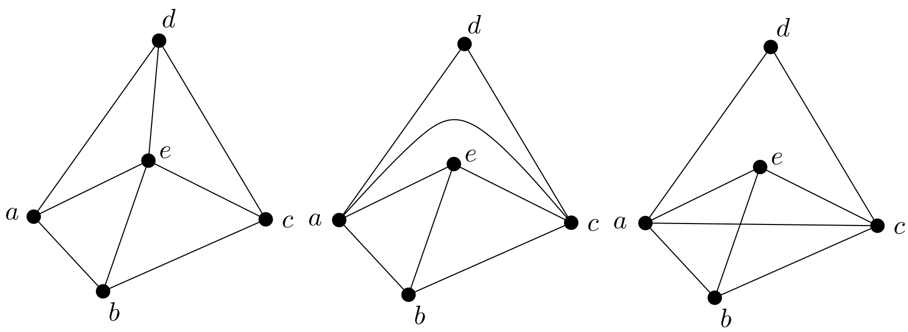
[1]A triangulation is a maximal planar simple graph.

The above setting of the problem is often referred to as the *combinatorial setting* of the problem since only a combinatorial embedding of the triangulation is specified[2]. Although many other settings of the problem have been studied in the literature (where the graph is embedded on different surfaces such as the torus), we continue with a review of the results in the geometric setting of the problem. In this setting there are a number of similarities as well as differences with the combinatorial setting. One of the differences is that the class of graphs studied is usually *near-triangulations* as opposed to triangulations. A near-triangulation is a triangulation with the property that one particular face (called the outer-face) need not be a triangle. In the geometric setting, the near-triangulation is embedded in the plane such that the vertices are points in the plane and the edges are straight-line segments with the property that two edges not sharing a common vertex do not intersect. Such an embedding is often referred to as a *planar straight-line embedding*. An edge flip is still a valid operation in the geometric setting (See Figure 1). Thus, similar questions have been studied. For example, Lawson [9] showed that given any two near-triangulations N_1 and N_2 embedded on the same n points in the plane, there always exists a finite sequence of edge flips that transforms the edge set of N_1 to the edge set of N_2. In Section 3, we present a brief review of some of the main results in the geometric setting of the problem followed by a discussion of some open problems.



(a) initial graph (b) valid combinatorial flip (c) invalid geometric flip

Figure 2: Edge $[de]$ can be flipped to $[ac]$ combinatorially but not geometrically.

Note that there is quite a discrepancy between the combinatorial setting of the problem and the geometric one. The discrepancy arises because not all combinatorially valid edge flips are geometrically valid (See Figure 2). In the combinatorial setting, Wagner [14] showed that *every* triangulation on n vertices can be transformed to every other triangulation via edge flips. On the other hand, in the geometric setting, Lawson [9] showed that only the near-triangulations that are defined on a specified point set can be attained via edge flips. For example, in the geometric setting, given a set of points in convex position, the only plane graphs that can be drawn without crossing are outer-planar graphs. This discrepancy has initiated a new line of investigation. Namely, does there exist a set of local operations, in addition to edge flips, that permits the enumeration of all n-vertex triangulations in the geometric setting. In Section 4, we present a review of some of these results followed by a discussion of some open problems.

2 Combinatorial Setting

The result that initiated the research on edge flips in triangulations is due to Wagner [14]. He proved that given a triangulation with n vertices, with a finite sequence of edge flips, one can transform this graph to any other triangulation on n vertices. The main idea behind Wagner [14]'s proof is that a finite sequence of edge flips allow one to transform a given triangulation to a canonical one. The canonical triangulation defined is one where there are two vertices in the triangulation that are adjacent to every other vertex of the triangulation (See Figure 3). The graph induced by these other vertices is a path and is referred to as the spine of the canonical triangulation. With this tool in hand, to transform

[2] In a combinatorial embedding of a planar graph, for each vertex of the graph, the clockwise order of the edges adjacent to the vertex is specified.

an n-vertex triangulation T_1 to a triangulation T_2, one first transforms T_1 into canonical form, then applies the flips to transform T_2 to canonical form in reverse order.

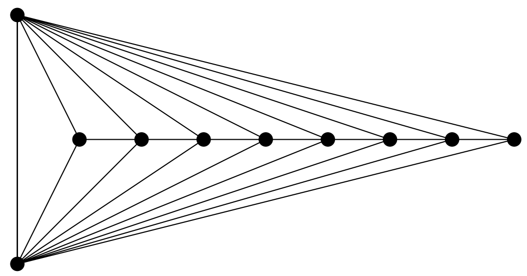


Figure 3: Wagner [14]'s canonical triangulation.

Given this seminal result, several natural questions about edge flips in triangulations leap to mind. Indeed, this result incited a flurry of activity in many different directions. We restrict our attention to results directly related to edge flips in the combinatorial setting. A careful analysis of Wagner [14]'s result reveals that the length of the edge flip sequence is at most $O(n^2)$ where n is the size of the triangulation. It is easy to see that there exist pairs of triangulations that require $\Omega(n)$ edge flips. Consider a triangulation having a vertex of linear degree and one where every vertex has constant degree. Since an edge flip only reduces the degree of a vertex by one, a linear number of edge flips is required to reduce the degree of a vertex from linear to constant. Komuro [8] proved that this bound is tight by showing that $O(n)$ edge flips suffice to transform any n-vertex triangulation to any other n-vertex triangulation. Mori *et al.* [10] currently have the best bound where they show that at most $6n - 30$ edge flips are sufficient. One can view this from a different perspective via the triangulation flip graph. The triangulation flip graph is a graph whose vertices are combinatorially distinct n-vertex triangulations and two vertices in the flip graph are adjacent provided that one edge flip is sufficient to transform the triangulation corresponding to one vertex to the triangulation corresponding to the other. Viewed from this perspective, Wagner [14] showed that the triangulation flip graph is connected and its diameter is $O(n^2)$. Komuro [8] showed that in fact that diameter is $O(n)$, and Mori *et al.* [10] reduced the constants to show that the diameter is at most $6n-30$. On the way to proving their result, Mori *et al.* [10] showed that given any n-vertex triangulation, at most $n - 4$ edge flips are sufficient to convert this to a 4-connected triangulation (which by a result of Tutte [13] is Hamiltonian), and $4n - 22$ edge flips are sufficient to convert any 4-connected triangulation to any other 4-connected triangulation.

Several interesting questions remain open: are there triangulations that require at least $n - 4$ edge flips to be converted to Hamiltonian or 4-connected? is $4n - 22$ the best upper bound for converting one 4-connected triangulation to another? is there a matching lower bound? is $6n - 30$ the best upper bound for converting one triangulation to another? Can one find matching upper and lower bounds? To date, all of the bounds are proven by showing how to transform a given triangulation into a canonical one (of some form). Clearly, this is not necessarily the best way for transforming a triangulation T_1 into T_2. For example, it may be that a single edge flip is sufficient to transform T_1 into T_2 but by going via a canonical triangulation, $O(n)$ flips are performed. Thus, is it possible to efficiently compute the smallest number of flips sufficient to transform a given triangulation T_1 into T_2 (i.e. without constructing the whole flip graph)? or can a sequence of flips be found whose length is related to (i.e. bounded by a constant or a $(1+\epsilon)$- approximation) the length of the shortest sequence? In terms of the flip graph, this is asking for the shortest path or an approximation of the shortest path between two vertices of the flip graph.

Another question of interest is the maximum number of edges that can be individually flipped (i.e. edges that are flippable) in a triangulation. Gao *et al.* [6] showed that every n-vertex triangulation has at least $n-2$ flippable edges and that there exist triangulations with at most $n-2$ flippable edges. The former result is proved by showing that every face has at least one edge that is flippable. The latter is through a simple construction where one starts with an initial triangulation T on m vertices and inserts a vertex inside each face of T, and completes the triangulation by joining the vertex to each of the three vertices of the face. The resulting n-vertex triangulation has only $n-2$ flippable edges. Note that separating triangles play a key role here. In a triangulation, every vertex of degree 3 is contained in a separating triangle. Therefore, triangulations with minimum degree at least 4 have more flippable edges. In fact, Gao *et al.* [6] show that every n-vertex triangulation with minimum degree at least 4 (for $n > 8$) has at least $2n + 3$ flippable edges. There exist triangulations that also achieve this bound, therefore, these bounds are tight. When viewed in terms of the flip graph, these questions are asking about the degree of a vertex. However, there is a subtle difference. Even if a triangulation has $n - 2$ flippable edges, it does not necessarily mean that flipping each of those edges leads to $n - 2$ combinatorially distinct triangulation. Therefore, it would be interesting to determine the maximum, minimum and average degree of a vertex in the flip graph.

In an n-vertex triangulation, since there are always a linear number of edges that can each be individually flipped, it seems natural to ask how many of these edges can be flipped simultaneously. This notion was introduced, albeit in the geometric setting, by Galtier *et al.* [5]. Given an n-vertex triangulation T and a subset S of edges, the operation of a *simultaneous flip* consists of flipping each of the edges in S to produce a distinct triangulation T'. Such a set S of edges is said to be *simultaneously flippable*. Sets of simultaneously flippable edges have a strong connection to the notion of a flippable edge but they are a different beast altogether. For example, it is possible for a set S of edges to be simultaneously flippable yet contain edges that are *not* individually flippable. It is also possible for every edge in a set S to be individually flippable but the set S itself *not* be simultaneously flippable. In this setting, the main question is how many simultaneous flips are sufficient to convert one n-vertex triangulation to another. The work on individually flippable edges trivially implies that $O(n)$ simultaneous flips are sufficient. The question is how much better can one do when one takes advantage of the ability to flip multiple edges at the same time. Bose *et al.* [3] showed that $O(\log n)$ simultaneous flips are sufficient to convert any n-vertex triangulation to any other. They showed that this bound is tight since there exist pairs of triangulations that require at least $\Omega(\log n)$ simultaneous flips to be converted to each other. The approach taken in Bose *et al.* [3] is to convert a triangulation into canonical form using simultaneous flips. As was shown by Mori *et al.* [10] for the case of single flips, Bose *et al.* [3] show that a few number of simultaneous flips are sufficient to convert a given triangulation into a 4-connected (Hamiltonian) one. In fact, they show that at most one simultaneous flip is sufficient. With respect to the maximum number of edges that can always be simultaneously flipped in an n-vertex triangulation, it is shown in [3] that at most $n - 2$ edges can ever be flipped simultaneously, that every triangulation has at least $(n - 2)/3$ edges that can be flipped simultaneously and that there exist triangulation where at most $6(n - 2)/7$ edges can be flipped simultaneously. A number of open problems remain. Can the gap between the lower bound of $(n-2)/3$ and upper bound of $6(n-2)/7$ be closed? Although asymptotically, the bounds on the number of simultaneous flips needed to convert any n-vertex triangulation to any other are tight, the constants are definitely not tight.

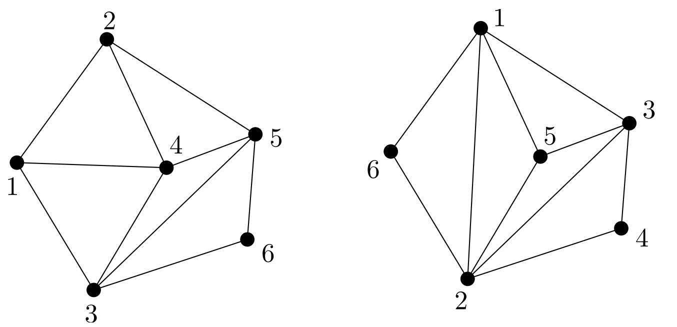


Figure 4: One flip is not sufficient to convert the left graph into the right graph in the labelled setting

So far, all of the results that have been discussed pertain to the unlabelled setting, that is given an initial triangulation, we wish to convert it to a final triangulation but are satisfied if the edge flips terminate with a triangulation that is isomorphic to the final triangulation. In the labelled setting, we are given an initial n-vertex triangulation and a final n-vertex triangulation defined on the same vertex set and we wish to bound the number of edge flips needed to convert the initial triangulation into the final one. Gao *et al.* [6] showed that $O(n \log n)$ edge flips are sufficient. Notice that if we transform both the initial and final triangulation into Wagner's canonical form without paying attention to vertex labels, then the problem in the labelled setting becomes one of sorting the vertices along the spine. This is essentially what Gao *et al.* [6] do in a divide-and-conquer fashion leading to the $O(n \log n)$

result. One interesting question is whether or not $\Omega(n \log n)$ is a lower bound or can this be achieved with $O(n)$ edge flips as in the unlabelled case? The results in Bose *et al.* [3] trivially imply an $O(n)$ bound for the number of simultaneous flips in the labelled setting. Can this be improved?

3 Geometric Setting

In the geometric setting, the graphs studied are straight-line planar embeddings of near-triangulations where vertices are points in the plane and edges are straight-line segments. The seminal result by Lawson [9] initiated the study of flips in the geometric setting. Lawson [9] showed that given any two near-triangulations N_1 and N_2 straight-line embedded on the same n points in the plane, there always exists a finite sequence of edge flips that transforms the edge set of N_1 to the edge set of N_2. The approach used by Lawson [9] is similar to that of Wagner [14] in that Lawson showed how to convert a given near-triangulation into canonical form using edge flips. The canonical form selected by Lawson is the Delaunay[4, 11] triangulation of the point set. Lawson [9] showed that $O(n^2)$ flips are sufficient to convert any n-vertex near-triangulation into the Delaunay triangulation of the same point set. Contrary to the situation in the combinatorial setting, Hurtado *et al.* [7] proved that this bound is tight by constructing a pair of n-vertex near-triangulations that require at least $\Omega(n^2)$ edge flips to convert one into the other. This leads to the question as to whether or not some of these bounds are sensitive to properties of the point set. In the case where the points are in convex position, Hurtado *et al.* [7] note that due to the bijection between triangulations of convex n-gons and binary trees with $n-2$ internal nodes, the result of Sleator *et al.* [12] implies that at most $2n-10$ edge flips are sufficient to convert any triangulation of a set of n points in convex position to any other triangulation of the same point set. If a set of n points has k convex layers[3], Hurtado *et al.* [7] show that $O(kn)$ edge flips are sufficient. For simple triangulated n-gons with k reflex vertices, $O(n + k^2)$ edge flips are sufficient. When studying the maximum number of edges that can be flipped in any near-triangulation of a set of n points in the plane, Hurtado *et al.* [7] prove that at least $\lceil (n-4)/2 \rceil$ edges are flippable. They show that this bound is tight by providing a construction (similar to the one described in Gao *et al.* [6]) that allows only $\lceil (n-4)/2 \rceil$ flippable edges. Several open questions remain in this area: can one find matching constants in the upper and lower bound on the number of edge flips? Can one find a matching lower bound for the case where the point set has k convex layers or is $O(kn)$ the correct asymptotic answer? Is there a class of graphs that can be reached in fewer edge flips? For example, in the combinatorial setting, fewer flips were needed to convert a given triangulation into a Hamiltonian one. Is the same true in the geometric setting? Is there always a sequence of $o(n^2)$ flips that allows one to convert any near-triangulation into a Hamiltonian one?

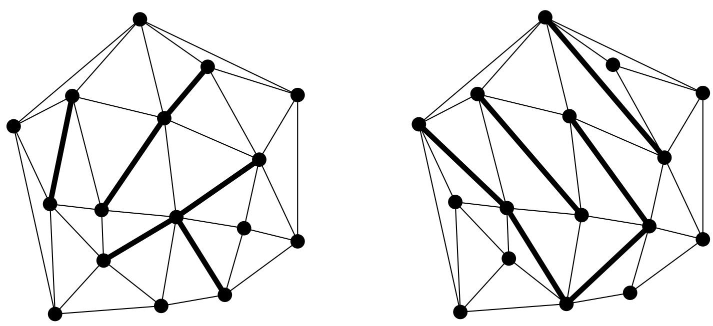


Figure 5: The darkened edges in the near-triangulation on the left are simultaneously flipped to give the triangulation on the right.

In an n-vertex near-triangulation, since there are always $\lceil (n-4)/2 \rceil$ edges that can be individually flipped, Galtier *et al.* [5] where the first to ask whether flipping several edges at the same time could help. They introduced the notion of a simultaneous geometric flip (this is similar to the notion of simultaneous flips discussed in the previous section. See Figure 5). Given an n-vertex near-triangulation

[3]the number of convex layers in a point set is the number of times the convex hull of a point set can be removed from the point set until the point set is empty.

T and a subset S of edges, the operation of a *simultaneous flip* consists of flipping each of the edges in S to produce a distinct near-triangulation T'. Such a set S of edges is said to be *simultaneously flippable*. Galtier *et al.* [5] showed that $O(n)$ simultaneous edge flips are sufficient to convert any n-vertex near-triangulation to any other near-triangulation on the same vertex set. They modified the construction in Hurtado *et al.* [7] to show that there exist pairs of near-triangulations that require $\Omega(n)$ simultaneous edge flips. For the restricted case where the points are in convex position, they showed that $O(\log n)$ simultaneous flips are sufficient and there are pairs of near-triangulations that require $\Omega(\log n)$ simultaneous flips. Finally, they showed that every near-triangulation on n points has at least $(n-4)/6$ edges that can be flipped simultaneously and that there exist triangulations that have at most $(n-4)/5$ edges that can be flipped simultaneously. A number of questions remain unsolved: Although asymptotically, the bounds on the number of simultaneous flips are tight both in the general case and the case where the points are in convex position, in neither case is the constant tight. Can the gap between the $(n-4)/6$ lower bound and $(n-4)/5$ upper bound be closed? Can a smaller number of simultaneous flips allow one to convert any n-vertex near triangulation into a Hamiltonian one? What happens if one restricts their attention to only Delaunay flips[4]?

4 Extensions of the Geometric Setting

As noted in the introduction, there is quite a discrepancy between the combinatorial setting of the problem and the geometric one. In the combinatorial setting, all the results are with respect to the class of triangulations whereas in the geometric setting, the transformations are restricted to a fixed point set. For example, Wagner [14] showed that *every* triangulation on n vertices can be transformed to every other triangulation via edge flips. On the other hand, in the geometric setting, Lawson [9] showed that only the near-triangulations that are defined on a specified point set can be attained via edge flips. This discrepancy initiated a new line of investigation. Namely, does there exist a set of local operations, in addition to edge flips, that permits the enumeration of all n-vertex triangulations in the geometric setting. In order to achieve this, it is essential to allow a point to be moved because given a set of n points in the plane, not all n-vertex triangulations can be straight-line embedded on the given point set. Abellanas *et al.* [1] defined a point move in an n-vertex triangulation embedded in the plane as simply the modification of the coordinates of one vertex of the graph. The point move is deemed valid provided that no edge crossings are introduced after the move. In this setting, Abellanas *et al.* [1] showed that $O(n)$ point moves and $O(n^2)$ edge flips are sufficient to transform any n-vertex triangulation embedded in the plane into any other n-vertex triangulation. Moreover, if the initial graph is embedded in an $n \times n$ grid, all point moves stay within a $5n \times 5n$ grid (i.e. the size of the coordinates in the move are bounded). Although Hurtado *et al.* [7] provide a pair of n-vertex near-triangulations that require $\Omega(n^2)$ edge flips to transform one into the other, this lower bound no longer holds in the presence of point moves. In fact, it can be shown that $O(n)$ point moves and edge flips are sufficient for this case. Therefore, the question becomes is there an $\Omega(n^2)$ lower bound on the number of edge moves required? If one removes the restriction on the size of the coordinates, Aloupis *et al.* [2] was able to show that with $O(n \log n)$ point moves and edge flips, one can convert any n-vertex straight-line embedded triangulation into any other. Is this best possible or can it be shown that a linear number of edge flips and point moves is sufficient? In the labelled setting, Abellanas *et al.* [1] showed that $O(n^2)$ point moves (with all moves restricted to the $5n \times 5n$ grid) and edge flips are sufficient. Aloupis *et al.* [2] proved that $O(n \log n)$ point moves and edge flips are sufficient when there are no restrictions on the size of the the coordinates. We conclude by asking whether or not these bounds are optimal?

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