On k-Guarding of Polygons

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Abstract

A polygon is called k-guardable if it is possible to find a collection G of points in the interior of the edges of P such that every point in P is visible from at least k elements of G, and such that no edge of P contains more than one element of G. In this paper we prove that every simple polygon can be 1-guarded using at most $\lfloor n/2 \rfloor$ guards, and that every polygon with 0 or 1 polygonal hole can be 2-guarded. We also prove that even though every polygon with holes is 1-guardable, for every $h \geq 2$ there exists a polygon with h holes that is not 2-guardable. Our proofs lead to linear time algorithms to find 1- and 2-guarding collections for polygons with 0 or 1 hole.

Introduction

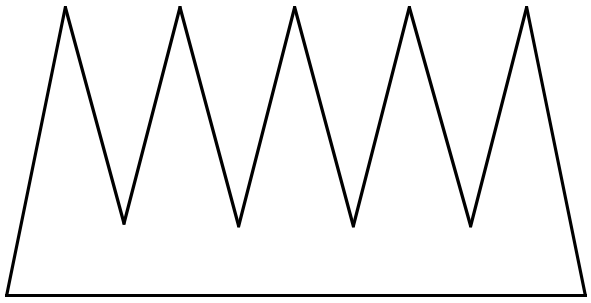
A *polygon* is an ordered sequence of three or more points $v_1, v_2, \ldots v_n$ in the plane called *vertices*, and n line segments $e_1, e_2, \ldots e_n$ called *edges*, where $e_i = \overline{v_i v_{1+i \bmod n}}$ for i = 1 to n-1. A polygon is said to be *simple* if no two non-consecutive edges intersect.

An enumeration of the vertices of a polygon P is called *counterclockwise* if the interior of P lies to the left of its edges. In this paper, all enumerations of polygon vertices will be done in counterclockwise order. A vertex v_i of P will be called convex if v_{i+1} lies to the left of the line joining v_{i-1} to v_i. An edge of the polygon will be called convex if both of its endpoints are convex.

Given a set S in the plane, a point p, and a real number $\varepsilon > 0$, we will denote by $N_\varepsilon(S, p)$ the set of all points of S whose distance to p is less than ε. Given an edge e of a polygon P, the *interior* halfplane of e is the halfplane H determined by the line containing e and such that, for every $\varepsilon > 0$, and every point p in the interior of e, the set $H \cap N_\varepsilon(P, p)$ contains points in the interior of P.

Let P be a polygon with h holes and n vertices. Two points p, q of P are *visible* if the line segment c joining p to q does not intersect the exterior of P. The points p and q are *clearly visible* if no point in the interior of c belongs to the boundary of P. We say that P is *k-guardable* if there exists a set G of points that belong to the interior of edges of P such that no edge of P contains more than one element of G, and such that every point of P is visible from at least k elements of G. A different kind of k-guardability has been previously studied by Pesant [Pes89].

The following question was asked by A. Lubiw at the open problem session of the Fourth Canadian Conference on Computational Geometry: For which values of k is every simple polygon k-guardable? It has been observed by T. Shermer that *comb* polygons [Chv75, O'R87] are not 3-guardable; such a polygon is shown in Figure 1. In this paper we prove that every simple polygon with n vertices can be 2-guarded using at most n points, and 1-guarded using at most $\lfloor n/2 \rfloor$ guards. These bounds are tight up to an additive constant; in both cases comb polygons realize the lower bound. For polygons with holes, we prove that every such polygon is 1-guardable, and that every polygon with one hole is 2-guardable. We also exhibit polygons with h holes that are not 2-guardable for every $h \geq 2$.



A polygon which is not 3-guardable

Figure 1.

2. Two-Guarding Simple Polygons.

In this section, we consider the problem of 1-guarding and 2-guarding simple polygons. A *chord* of P is a line segment whose endpoints belong to the boundary of P, that does not intersect the exterior of P, and whose interior does not intersect the boundary of P.

The *visibility polygon* from a point a of P, denoted by P_a, is the set of all points of P that are visible from a. If an edge e of P_a is a chord of P, the enpoint x of e furthest from a will be called a *target* point of P_a. The endpoint of e closest to a, which is always a reflex vertex of P, will be denoted by $v_a(x)$.

We are now ready to prove our first result, namely that every simple polygon can be 2-guarded. We will in fact prove a slightly stronger result which will be used in the next section as well.

Theorem 1 : Let P be a simple polygon, e_j be an edge of P at least one of whose endpoints is convex, and ε_{j-1}, ε_{j+1} be non-negative real numbers. There exists a set G = $\{g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_n\}$ of points of P such that

1. Guard g_i belongs to edge e_i of P;
2. Guard g_i lies in the interior of e_i whenever $| j - i | \neq 1$;
3. If e_j is a convex edge of P, then the quadrilateral $g_{j-1}v_jv_{j+1}g_{j+1}$ is a convex subpolygon of P.
4. If $| j - i | = 1$ and v_i is a convex vertex of P, then the distance between g_i and v_i is strictly positive unless $\varepsilon_i = 0$, and is at most ε_i.

Proof : Let us assume by symmetry that v_j is a convex vertex of P. If v_{j+1} is reflex, let $\varepsilon = \varepsilon_{j-1}$. Otherwise, since P is simple, there exists $\varepsilon \leq \min \{\varepsilon_{j-1}, \varepsilon_{j+1}\}$ such that every point of $N_\varepsilon(e_{j-1}, v_j)$ sees every point of $N_\varepsilon(e_{j+1}, v_{j+1})$. Let us place a guard at a point a of $N_\varepsilon(e_{j-1}, v_j)$, and let $I = \{ i \mid$ the interior of e_i intersects $P_a \}$. Finally, for each $i \in I$, let $s_i = \overline{b_ic_i}$ be the subset of e_i that belongs to P_a, let x_i be an arbitrary point of s_i if $i \neq j+1$, and let x_i be a point of s_i whose distance from v_i is at most ε_{j+1} if $i = j+1$ (such a point always exists if $j+1 \in I$, since then v_{i+1} is convex , and so $a \in N_\varepsilon(e_{j-1}, v_j)$).

We can construct Q_a from P_a by removing, for each $i \in I$, the triangles $\Delta x_i b_i v_a(b_i)$ and $\Delta x_i c_i v_a(c_i)$, provided that $v_a(b_i)$ and $v_a(c_i)$ exist respectively. Figure 2 illustrates this procedure.

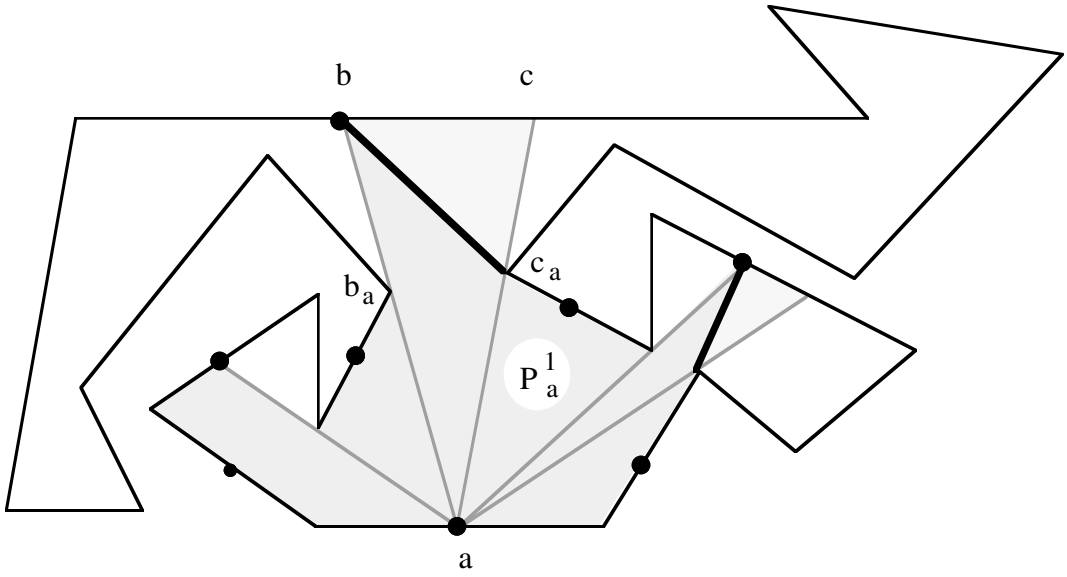


Figure 2.

Now, let $G_a = \{a\} \cup \{x_i \mid i \in I\}$. Every point of Q_a is visible from two guards in G_a since every subset in the partition of Q_a induced by joining a to every vertex of Q_a contains two guards.We now show how to guard each simply connected component of $P \setminus Q_a$. Let Q be such a component, and let c be $Q \cap Q_a$. Two cases need to be considered.

1. **(c joins two vertices of P) :** c is a convex edge of Q, and so a 2-guarding G_Q of Q that does not contain any guard in the interior of c can be computed recursively with $\varepsilon_{j-1} = \varepsilon_{j+1} = 1$.

2. **(c joins a reflex vertex of P to an interior point p of an edge of P)** : There is already a guard on p, so we can compute recursively a 2-guarding G_Q of Q that does not contain any guard in the interior of c with $\varepsilon_{j-1} = 1$ and $\varepsilon_{j+1} = 0$ (if p is the counterclockwise endpoint of c) or $\varepsilon_{j-1} = 0$ and $\varepsilon_{j+1} = 1$ (if p is the clockwise endpoint of c)

Hence, $G_a \cup \{ G_Q \mid Q$ a simply connected component of $P \setminus Q_a \}$ is a 2-guarding of P that satisfies the conditions of the theorem.

QED.

Our first two main results are corollaries of Theorem 1 and are easily derived from it.

Corollary 1: Every simple polygon can be 2-guarded.

Proof: Let P be a simple polygon, and let v_j be a convex vertex of P. Let Q be the polygon obtained from P by duplicating v_j and adding a zero-length edge e between the two copies of v_j. Every 2-guarding of Q in which no guard is placed on e and obtained with $\varepsilon_{j-1} = \varepsilon_{j+1} = 1$ is a 2-guarding of P.

QED.

A 2-guarding obtained using the procedure described in the proof of Theorem 1 is illustrated in Figure 3.

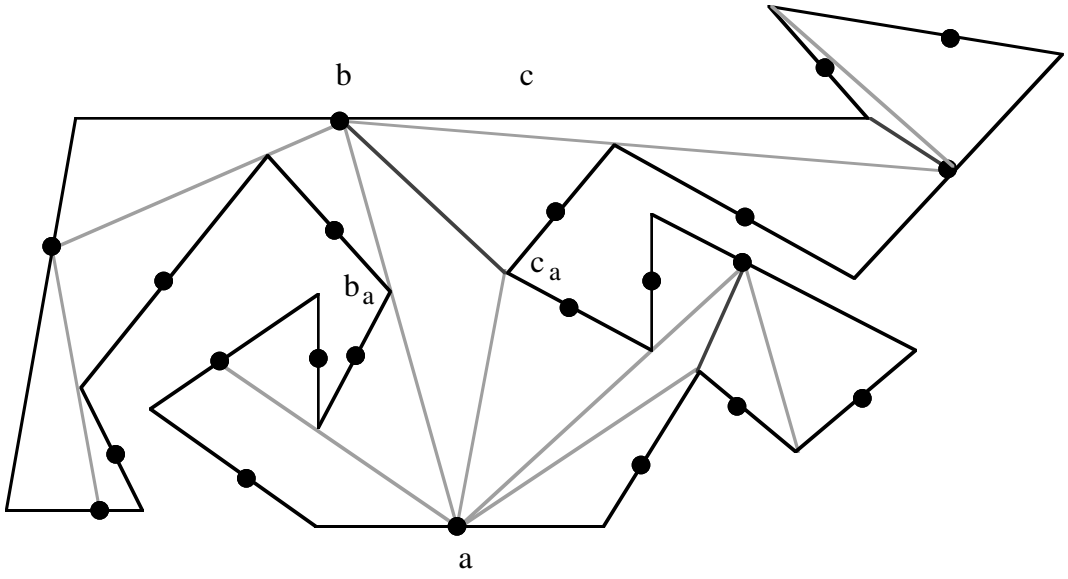


Figure 3.

Corollary 1: Every simple polygon can be 2-guarded.

Proof: Let P be a simple polygon for which we want to find a 2-guarding, and let v_j be a convex vertex of P. Meisters [Mei75] showed that such a v_j always exists. Let Q be the polygon obtained for P by duplicating v_j and adding a zero-length edge between the two copies of v_j. Every 2-guarding of Q obtained with $\varepsilon_{j-1} = \varepsilon_{j+1} = 1$ is a 2-guarding of P.

QED.

Corollary 2: $\lfloor n / 2 \rfloor$ guards are always sufficient and sometimes necessary in a 1-guarding of a simple polygon.

Proof: The family of comb polygons similar to the polygon shown in Figure 1 demonstrates that $\lfloor n / 2 \rfloor$ guards are sometimes required. Let us now show that $\lfloor n / 2 \rfloor$ guards are always sufficient.

Let us assume that *col* = 1 when we first call the procedure described in the proof of Theorem 1. In each call, we assign the current value of *col* to a, and then replace *col* by *3 - col*. All guards subsequently placed during this invocation of the procedure will then be colored using the new value of *col*. We note that, if $\varepsilon_{j+1} = 0$, then we want a to be the guard placed on e_{j+1} instead of the guard placed on e_{j-1}. The remainder of the procedure is then executed symmetrically.

By our colouring rule, it follows that every point of P is seen by at least one guard that was assigned colour 1, and one that was assigned colour 2. One of the two colours occurs at most $\lfloor n / 2 \rfloor$ times, and so the set of guards of that color is the required 1-guarding of P.

QED.

2. Polygons With Holes.

In the previous section, we proved that every simple polygon without holes is 2-guardable. In this section we study the case in which polygons are allowed to have holes. We first consider 2-guardability.

Theorem 2: For each h ≥ 2, there exists a polygon with h holes that is not 2-guardable.

Proof : Let us first consider the case in which h = 2. Let P be the polygon with holes shown in Figure 4, S be the set of six points in the interior of P denoted with small round circles and E be the set of edges { e_1, e_2, …, e_{11}}. It is easy to see that the only edges of P containing interior points that can see one or more elements of S are those in E. Moreover, no interior point of any of these edges can see more than one element of S. Therefore in order to 2-guard the elements of S, we would need to choose twelve different points in the interior of the edges in E. Since E contains only eleven edges, it follows no 2-guarding of S (and hence of P) exists. For h > 2, each new hole added to P (following the obvious pattern) adds two points to S, but only 3 edges to E, and so our argument still holds.

QED.

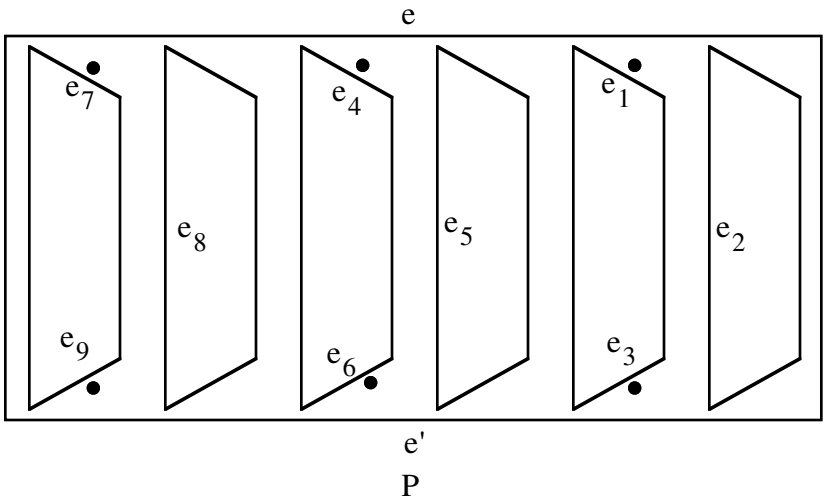


Figure 4.

In view of Theorems 1 and 2, one could wonder whether polygons with exactly one hole are 2-guardable or not. Before answering this question, we need to prove a few lemmas about a specific kind of polygons which are called *funnels*. A *funnel* is a simple polygon P that has a special vertex v_k (1 < k < n), and such that v_1, v_k and v_n are the only convex vertices of P.

Let F be a funnel polygon, let C_1 denote the reflex chain v_1, …, v_k, and let C_2 denote the reflex chain v_k, …, v_n. Let $v_0 = v_{n+1}$ be the point of intersection of the edge v_1v_n of F with the bisector of the angle $\angle v_{k-1}v_kv_{k+1}$. We note that v_0 is clearly visible from every vertex of F except v_1 and v_n. For i = 1 to k-1, let F_i denote the subset of F that lies on the same side of the chord $\overline{v_0v_{i-1}}$ as v_k. For i = k+1 to n, let F_i denote the subset of F that lies on the same side of the chord $\overline{v_0v_{i+1}}$ as v_k. These definitions are illustrated in Figure 5a.

Lemma 1: Let Q_{i-1} be a simple polygon whose intersection with a funnel F is the edge $\overline{v_{i-1}v_i}$ of C_1, let Q be $Q_{i-1} \cup F$, and let e be the edge of Q clockwise from v_i. There exists $\delta(i-1) > 0$ such that every point of F_i that sees v_i and belongs to the interior halfplane of e is visible from every point of $N_{\delta(i-1)}(e, v_i)$.

Proof : Since v_0 is clearly visible from v_i, there exists $\delta > 0$ such that every point of $N_\delta(e, v_i)$ lies on the same side of the line through v_0 and v_{i-1} as v_i. Since Q_{i-1} is simple, there exists $\delta' > 0$ such that every point of $N_{\delta'}(e, v_i)$ sees every point of the line segment $\overline{v_{i-1}v_i}$. Let $\delta(i-1) = \min \{\delta, \delta'\}$, and consider a point x of F_i that lies in the interior halfplane of e, and a point y of $N_{\delta(i)}(e, v_i)$. We need to show that x sees y.

Consider the line segment $c = \overline{xz}$, where initially $z = v_i$. Suppose that we move the endpoint z of c that belongs to e towards y until one of the following events occurs:

1. z reaches y;
2. c intersects the exterior of Q;

It suffices to show that the second event cannot occur before the first. Suppose that this is the case. As c is about to intersect the exterior of Q, there is a reflex vertex v of Q in the interior of c. Since $\delta(i-1) \le \delta$, and since x belongs to F_i, it follows that $v \ne v_{i-1}$. Hence v belongs to Q_{i-1}. However, this is impossible since $\delta(i-1) \le \delta'$.

Therefore, $\overline{xy}$ is contained in Q, and so x sees y. This completes the proof of the lemma.

QED.

A symmetric lemma holds in the case of an edge $\overline{v_iv_{i+1}}$ that belongs to C_2. We will state it without proof.

Lemma 2: Let Q_i be a simple polygon whose intersection with a funnel F is the edge $\overline{v_i v_{i+1}}$ of C_2, let Q be $Q_i \cup F$, and let e be the edge of Q counterclockwise from v_i. There exists $\delta(i) > 0$ such that every point of F_i that sees v_i and belongs to the interior halfplane of e is visible from every point of $N_{\delta(i)}(e, v_i)$.

The proof of Lemma 1 yields two symmetric corollaries which will also prove to be useful, and concern the case where there is no Q_i.

Corollary 3: Let F be a funnel. Every point of F_i that belongs to the interior halfplane of an edge $\overline{v_{i-1} v_i}$ of C_1 and sees v_i sees every point of $\overline{v_{i-1} v_i}$.

Corollary 4: Let F be a funnel. Every point of F_i that belongs to the interior halfplane of an edge $\overline{v_i v_{i+1}}$ of C_2 and sees v_i sees every point of $\overline{v_i v_{i+1}}$.

For i = 1 to k-1, let F_i' be the subpolygon of F whose vertices are v_i, v_{i+1}, $v_{l(i)}$, $v_{l(i)+1}$, …, $v_{m(i)}$ where $v_{l(i)}$, …, $v_{m(i)}$ are the vertices of C_2 that are clearly visible from v_i (with $l(i) \leq m(i)$). For i = k + 1 to n, let F_i' be the subpolygon of F whose vertices are v_{i-1}, v_i, $v_{l'(i)}$, …, $v_{m'(i)}$, where $v_{l'(i)}$, …, $v_{m'(i)}$ are the vertices of C_1 that are clearly visible from v_i (with $l'(i) \leq m'(i)$). These are shown in Figure 5b. We now present a characterization similar to that of Lemmas 1 and 2.

Lemma 3: Let Q_i be a simple polygon whose intersection with a funnel F is the edge $\overline{v_i v_{i+1}}$ of C_1, let Q be $Q_i \cup F$, and let e be the edge of Q counterclockwise from v_i. There exists $\varepsilon(i) > 0$ such that every point of F_i' that belongs to the interior halfplane of e is visible from every point of $N_{\varepsilon(i)}(e, v_i)$.

Proof : The lemma clearly holds if no point of F_i' belongs to the interior halfplane of e, so let us assume otherwise. Let w be the point of C_2 closest to $v_{m(i)}$ that belongs to the interior halfplane of e. Since w is clearly visible from v_i and belongs to the interior halfplane of e, there exists $\varepsilon > 0$ such that every point of $N_\varepsilon(e, v_i)$ is visible from w. Since $v_{l(i)}$ is clearly visible from v_i and belongs to the interior halfplane of e by assumption, there exists $\varepsilon' > 0$ such that every point of $N_{\varepsilon'}(e, v_i)$ is visible from $v_{l(i)}$. Since Q is simple, there exists $\varepsilon'' > 0$ such that every point of $N_{\varepsilon''}(e, v_i)$ sees every point of the line segment $\overline{v_i v_{i+1}}$. Let $\varepsilon(i) = \min \{\varepsilon, \varepsilon', \varepsilon''\}$, and consider a point x of F_i' that lies in the interior halfplane of e, and a point y of $N_{\varepsilon(i)}(e, v_i)$. We need to show that x sees y.

Consider the line segment $c = \overline{xz}$, where initially $z = v_i$. Suppose that we move the endpoint z of c that belongs to e towards y until one of the following events occurs:

1. z reaches y;
2. c intersects the exterior of Q;

It suffices to show that the second event cannot occur before the first. Suppose that this is the case. As c is about to intersect the exterior of Q, there is a reflex vertex v of Q in the interior of c. Since $\varepsilon(i) \leq \varepsilon$, v does not belong to C_2. Furthermore, since $\varepsilon(i) \leq \varepsilon'$, v is not v_{i+1} either. Hence v belongs to Q_i. However, this is impossible since $\varepsilon(i) \leq \varepsilon''$.

Therefore, $\overline{xy}$ is contained in Q, and so x sees y. This completes the proof of the lemma.

QED.

Once again, a symmetric version of this lemma holds for the case where $\overline{v_i v_{i+1}}$ belongs to C_2. We state it without proof.

Lemma 4: Let Q_{i-1} be a simple polygon whose intersection with a funnel F is the edge $\overline{v_{i-1} v_i}$ of C_2, let Q be $Q_{i-1} \cup F$, and let e be the edge of Q clockwise from v_i. There exists $\varepsilon(i-1) > 0$ such that every point of F_i' that belongs to the interior halfplane of e is visible from every point of $N_{\varepsilon(i-1)}(e, v_i)$.

The proof of Lemma 3 yields two symmetric corollaries which will also prove to be useful, and concern the cases where there is no Q_i.

Corollary 5: Let F be a funnel. For each edge $\overline{v_i v_{i+1}}$ of C_1, there exists $\varepsilon(i) > 0$ such that every point of F_i' that belongs to the interior halfplane of $\overline{v_i v_{i+1}}$ and sees v_i is visible from every point of $N_{\varepsilon(i)}(\overline{v_i v_{i+1}}, v_i)$.

Corollary 6: Let F be a funnel. For each edge $\overline{v_{i-1} v_i}$ of C_2, there exists $\varepsilon(i-1) > 0$ such that every point of F_i' that belongs to the interior halfplane of $\overline{v_{i-1} v_i}$ and sees v_i is visible from every point of $N_{\varepsilon(i)}(\overline{v_{i-1} v_i}, v_i)$.

Having proved the existence of $\delta(i)$ and $\varepsilon(i)$ in the cases where there is a single polygon "dangling" from an edge of F, we now show that we can find such an ε in a more complicated situation as well.

Lemma 5: Let I be a subset of { 1, 2, …, n-1 }, and S be a set $\{ Q_i \mid i \in I \}$ of simple polygons that satisfy the following conditions :

1. $Q_i \cap F = \overline{v_i v_{i+1}}$;
2. $Q_i \cap Q_j = v_{i+1}$ whenever $j - i = 1$;
3. $Q_i \cap Q_j = \varnothing$ whenever $| j - i | > 1$.

Finally, let $Q = F \cup (\cup_{i \in I} Q_i)$. There exists $\varepsilon > 0$ such that, for every edge e of Q incident upon a vertex v_i of F ($i \neq k$), every point of $F_i \cap F_i'$ that lies in the interior halfplane of e is visible from every point of $N_\varepsilon(e, v_i)$.

Proof : Let $\varepsilon = \min \{\min_{i \in I} \{\delta_i\}, \min_{i \neq k} \{\varepsilon_i\}\}$, and consider a point x of $F_i \cap F_i'$. Let us first assume that $i \in I$. Suppose that v_i belongs to C_1 and that e lies clockwise from v_i. Since $\varepsilon \leq \delta_{i-1}$, it then follows from Lemma 1 that x is visible from every point of $N_\varepsilon(e, v_i)$. A similar analysis using Lemmas 2, 3 and 4 shows that x will also be visible from every point of $N_\varepsilon(e, v_i)$ in the other three cases. Let us now assume that $i \notin I$. Suppose that v_i belongs to C_1 and that e lies clockwise from v_i. It follows from Corollary 3 that x is visible from every point of $N_\varepsilon(e, v_i)$. A similar analysis using Corollaries 4, 5 and 6 shows that x will also be visible from every point of $N_\varepsilon(e, v_i)$ in the other three cases.

QED.

We are now ready to prove that every polygon with 1 hole admits a 2-guarding, and to show how such a 2-guarding can be obtained.

Theorem 3: Every polygon with one hole is 2-guardable.

Proof : Let P be a polygon, let H be the polygonal hole contained in P, let $v_1, \ldots, v_s$ be a counterclockwise enumeration of the vertices of P, let $w_1, \ldots, w_t$ be a clockwise enumeration of the vertices of H, and let e_i be the edge of P that joins v_i to $v_{(i \bmod s)+1}$ for i = 1 to s.

Let w_j be a vertex on the convex hull of H, and let x be a point on the boundary of P that satisfies the following three conditions.

1. x is visible from w_j.
2. x is not a vertex of P.
3. The angles $\angle x w_j w_{j+1}$ and $\angle x w_j w_{j-1}$ are convex.

Such a point can be chosen by considering the subset S of P_{w_j} that lies in the interior halfplanes determined by both edges of the convex hull of H that are incident upon w_j. All points of S satisfy conditions 1 and 3, and at least one point on the boundary of S satisfies condition 2 as well. Let us assume without loss of generality that x belongs to e_i. Finally, let P' be the simple polygon whose vertices are $x v_{i+1} \ldots v_n v_1 \ldots v_i x w_j w_{j+1} \ldots w_t w_1 \ldots w_j$. This construction is illustrated in Figure 6.

Suppose that we now position a guard at x and apply the procedure described in the proof of Theorem 1 with $\varepsilon_{j-1} = 0$ and $\varepsilon_{j+1} = 1$ to obtain a 2-guarding G of P' (also shown in Figure 6) in which no guard is placed in the interior of $\overline{w_j x}$. G is not a 2-guarding of P for two reasons :

1. There will be two guards on edge e_i.
2. There will be a guard in the interior of the chord $\overline{w_j x}$ of P;

We need to show how to modify our procedure to obtain a 2-guarding of P.

Consider the first invocation of the procedure described in the proof of Theorem 1 during which a guard is to be placed on one of the line segments $\overline{w_j x}$, $\overline{x v_{i+1}}$. We will modify the manner in which this step is performed. Let G_0 be the set of all guards positioned on edges of P during the other invocations of the procedure.

Let Q be the subpolygon of P that is to be covered, and let us assume that a sees a point of $\overline{w_j x}$ (the case in which a sees no point of $\overline{w_j x}$ but one or more points of $\overline{x v_{i+1}}$ is symmetric and will not be discussed). Let F be the subpolygon of Q whose boundary consists of SP(x, a), SP(a, w_j) and $\overline{w_j x}$, as shown in Figure 7.

Let $u_1 = a, u_2, \ldots, u_{k-1}, u_k = a, u_{k+1}, \ldots, u_{m-1}, u_m = w_j$ be the vertices of F. We note that F is a funnel (the edges $\overline{u_{k-1} u_k}$ and $\overline{u_k u_{k+1}}$ may be collinear) and that every reflex vertex of F is a reflex vertex of P. Let $I = \{ i \mid \overline{u_i u_{i+1}}$ is a chord of P $\}$, and for each $i \in I$, let Q_i be the subpolygon of Q that joins u_i to u_{i+1}. Finally, let ε_0 be the value of ε whose existence is guaranteed by Lemma 5.

Consider a component Q_i of $Q \setminus F$. It follows from Theorem 1 that there is a 2-guarding G_i of Q_i such that

1. No guard has been placed on the edge of Q_i that is a chord of Q.

2. The guard already positioned at a (x) belong to G_i if a (x) belongs to Q_i.

3. The distance between a guard placed on an edge of Q_i incident upon a vertex v of F and the vertex v is at most ε_0.

Finally, let us place a guard g_i on each edge $e = \overline{u_i u_{i+1}}$ of F that is an edge of P and does not contain a or x, so that the distance between g and u_i (if $e \in SP(a, x)$) or u_{i+1} (if $e \in SP(a, w_j)$) is at most ε_0. Let G' be the set of all such guards.

We now claim that $G = G_0 \cup G' \cup (\cup_{i \in I} G_i)$ is a 2-guarding of P. We only need to show that F is 2-guarded. Indeed, let T be an arbitrary triangulation of F, let t be a triangle of T, and let y be a point in the interior of t. Since F is a funnel, t has two vertices $u_r \in C_1$ and $u_s \in C_2$, so that $r \neq k$ and $s \neq k$. We note that $t \subseteq (F_r \cap F_r') \cap (F_s \cap F_s')$. We now show that there is a guard that sees y placed on an edge of Q incident upon v_r. Several cases need to be considered.

1. **$v_r = a$ or $v_r = x$** : The guard placed on v_r sees y.

2. **$v_r \neq a$ and $v_r \neq x$** : The point y belongs to the interior halfplace of one of the edges of Q that is incident upon v_r. Hence it follows from Lemma 5 that the guard placed on that edge sees y.

Similarly, we prove that there is a guard that sees y placed on an edge of Q incident upon v_s. Once again several cases need to be considered.

1. **$v_s = a$** : The guard placed on v_s sees y.

2. **$v_s = w_j$** : By our choice of x, y belongs to the interior halfplane of the edge of Q counterclockwise from w_j. Hence it follows from Lemma 5 that the guard placed on that edge sees y.

3. **$v_r \neq a$ and $v_r \neq w_j$** : The point y belongs to the interior halfplace of one of the edges of Q that is incident upon v_r. Hence it follows from Lemma 5 that the guard placed on that edge sees y.

Since v_r and v_s do not belong to the same reflex chain of F, the guards placed on the edges of Q incident upon v_r and v_s are distinct, and so every point of F is seen by two distinct guards as required. Consequently G is a 2-guarding of P.

QED.

Before we proceed to prove that every polygon with holes is 1-guardable, we recall the following result on visibility.

Lemma 6: Let $S=\{\lambda_1,...,\lambda_n\}$ be a collection of disjoint line segments and p a point of the plane from which there is a ray that does not intersect any element of S. Then S contains at least one line segment λ_i that is completely visible from p.

A proof of this Lemma can be obtained from results presented in [Foldes, Rival and Urrutia]. It is easy to see that p induces an order relation "<" in S as follows:

i) We say that λ_a blocks λ_b (denoted by $\lambda_a \rightarrow \lambda_b$) if there is a point q in λ_b such that the line segment joining p to q intersects λ_a.
ii) We now say that $\lambda_a < \lambda_b$ if $\lambda_a \rightarrow \lambda_b$ or there is a chain of elements $\lambda_a = \lambda_1 \rightarrow \lambda_2 ... \rightarrow \lambda_k = \lambda_b$.

In the language of [Foldes] "<" is a light source order. Thus the element λ_i whose existence follows from Lemma 6 is nothing else than a minimal element of the order relation "<" on S.

Theorem 4: Every polygon with holes is 1-guardable.

Proof of Theorem 4: Let P be a polygon with holes. Without loss of generality, assume that the difference between the y-coordinate of any two vertices of P is at least ε for some $\varepsilon > 0$.

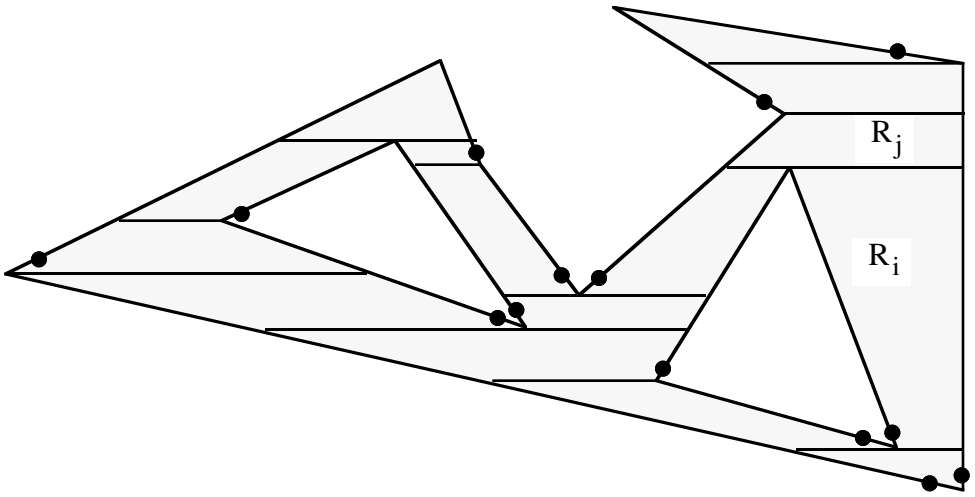


Figure 8.

For every vertex v of P consider the longest line segment contained in P that is parallel to the x-axis and contains v. These line segments partition P into a collection $R=\{R_1,....,R_m\}$ of convex polygons whose interiors are disjoint. For every edge e of P, place a guard in its interior at distance at most $\lfloor \varepsilon/2 \rfloor$ from e's lower endpoint, as illustrated in Figure 8. We claim that these points 1-guard P. In order to prove our claim we observe that if the boundary of a region R_i of R intersects the interior of an edge e and also contains its lower endpoint, then it contains the guard assigned to e. Suppose then that an element R_j of R does not contain a guard in its boundary and consider a point p in R_j. If p belongs to a line segment contained in P and that contains an edge e of P, then the guard assigned to e guards p. Suppose then that this is not the case. Using the horizontal line through p cut the polygon P in two parts and delete the part of P above it. At every remaining vertex w of P, cut away a sufficiently small segment from each edge of P incident upon w, or the remaining segment of an edge of P (See Figure 9). Notice that we get a disjoint family of line segments for which p is externally visible. By Lemma 2, one of these segments, say e', is completely visible from p. Since p is in the interior of P, it follows that p sees the side of e' facing towards the interior of P, and thus p is visible from the guard assigned to the edge of P that contains e'.

QED.

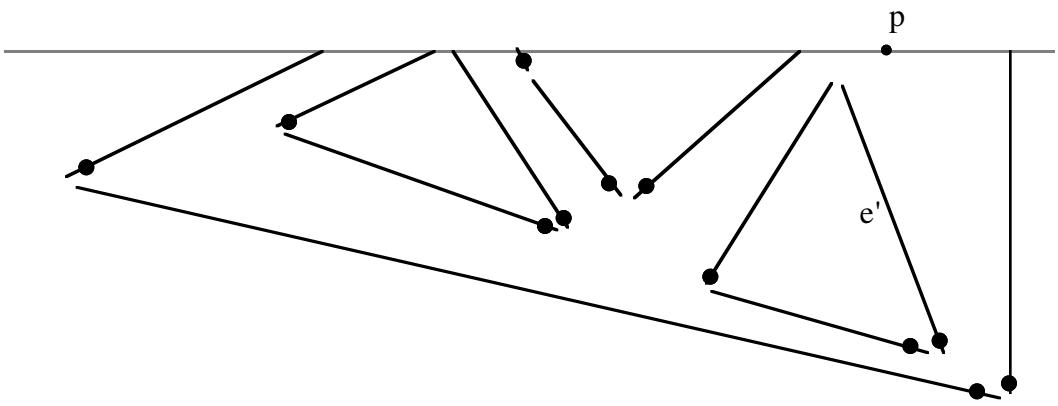


Figure 9.

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