Equitable subdivisions of polygonal regions

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Abstract. We prove a generalization of the Ham-Sandwich Theorem. Specifically, let P be a simple polygonal region containing $|R| = kn$ red points and $|B| = km$ blue points in its interior with $k \geq 2$. We show that either there exists two points a, b on the boundary of P such that the geodesic path between a and b forms an equitable 2-cut or there exists a point x inside P and three points a, b, c on the boundary such that the geodesics ax, bx and cx form an equitable 3-cut. A 2-cut is equitable if there exist integers $0 < k_1, k_2 < k$, with $k = k_1 + k_2$, and a partition of P into two relatively-convex regions P_1, and P_2, where each P_i contains exactly $k_i n$ red points and $k_i m$ blue points. Similarly, a 3-cut is equitable if there exist integers $0 < k_1, k_2, k_3 < k$, with $k = k_1 + k_2 + k_3$, and a partition of P into three relatively-convex regions P_1, P_2, and P_3, where each P_i contains exactly $k_i n$ red points and $k_i m$ blue points. We provide an $O(N^2 \log N)$ time algorithm for computing an equitable 2-cut or 3-cut where $N = |R| + |B| + |P|$.

1 Introduction

The two-dimensional case of the discrete *ham-sandwich theorem* (cf. [10]) states that, for finite point sets R (red points) and B (blue points) in $\mathbb{R}^2$, there always exists a line L (a *ham-sandwich cut*) such that each of the two open halfplanes defined by L contains at most half of the elements of both R and B. The discrete ham-sandwich problem has attracted considerable attention from an algorithmic viewpoint, culminating in a randomized algorithm by Lo and Steiger [7], with linear expected running time, whose subsequent derandomization (cf. [?]) completely settled the asymptotic worst case complexity of the problem.

Higher dimensional and continuous (with points replaced by general mass distributions) versions of the ham-sandwich problem have also been well studied. Very recently Bose *et al* [3] proposed a new generalization of the discrete ham-sandwich problem in which the sets R and B are restricted to the interior of a simple polygon P and permissible cuts consist of P-*geodesics* (shortest paths in P) joining two boundary points of P. They describe an efficient randomized algorithm for finding a P-geodesic bipartition of P where both of the (open) parts contain at most half of the elements of both R and B. Their algorithm has

$O(N \log W)$ expected running time, where $N = |R| + |B| + |P|$ and W measures the non-convexity of P (two plus the number of reflex vertices), which they show is optimal in the algebraic computation tree model.

Suppose that $|R| = kn$ and $|B| = km$, for integers $k \geq 2$, and $n, m \geq 0$. A partition of $R \cup B$ into $k \geq 2$ sets is said to be *equitable* if each set contains exactly n elements of R and m elements of B. Assuming that the points in R and B are in general position, a ham-sandwich cut provides an equitable bipartition of the plane (i.e. it induces an equitable bipartition of $R \cup B$) into convex regions. Similarly a geodesic ham-sandwich cut provides an equitable bipartition of a polygon P into two relatively-convex regions. (A set $S \subseteq P$ is said to be *convex relative to* P, or simply *relatively-convex* when P is understood, if every P-geodesic joining points of S is contained entirely within S.)

It is not hard to see that non-simple polygons, even polygons with a single slit in their interior, may not admit equitable bipartitions into relatively convex regions. This precludes the most natural further generalization of ham-sandwich partitions.

In 1998, Kaneko and Kano [6] conjectured that the two-dimensional discrete ham-sandwich theorem can be generalized to the existence of equitable convex k-partitions of the plane (partitions of the plane into k convex regions that induce an equitable partition of $R \cup B$), for any $k \geq 2$. The Kaneko-Kano conjecture was proved independently by Ito *et al* [5], for the case $k = 3$, Sakai [8] and Bespamyatnikh *et al* [2]. Only [2] considered efficient constructions of equitable convex k-partitions, giving an algorithm with worst-case time $O(N^{4/3} \log^3 N \log k)$, where $N = |R| + |B|$.

In light of the above, it is natural to ask if there is an analog of the notion of equitable convex k-partitions of the plane, when the sets R and B (as well as partition boundaries) are restricted to the interior of a simple polygon P. Specifically, does P always admit a k-partition into relatively convex regions that induce an equitable partition of $R \cup B$? In this paper we provide an affirmative answer to this question by providing a non-trivial generalization of the construction of Bespamyatnikh *et al* [2]. Specifically, we prove the following:

Theorem 1. *Let $k \geq 2$ and let R and B be point sets in general position inside a simple polygonal region P, consisting of kn red points and km blue points, respectively. Then either (i) there exist integers $0 < k_1, k_2 < k$, with $k = k_1 + k_2$, and a partition of P into two relatively-convex regions P_1, and P_2, where each P_i contains exactly $k_i n$ red points and $k_i m$ blue points, or (ii) there exist integers $0 < k_1, k_2, k_3 < k$, with $k = k_1 + k_2 + k_3$, and a partition of P into three relatively-convex regions P_1, P_2, and P_3, where each P_i contains exactly $k_i n$ red points and $k_i m$ blue points.*

When $k = 2$ only case (i) is applicable and Theorem 1 is equivalent to the geodesic ham-sandwich (existence) theorem of Bose *et al* [3]. For $k \geq 3$, Theorem 1 can be applied recursively to the regions P_i with $k_i > 1$, resulting in a k-partition of P into relatively convex regions, each of which contains exactly n red and m blue points, as desired. Figure 1 illustrates a relatively-convex equitable 3-partition of a simple polygon containing 9 red (empty) and 3 blue (solid) points.

It is not hard to see that (even in the unconstrained setting) partitions of type (i) (reducing bipartitions) do not always exist. For partitions of type (ii), we can assume the partition of P cuts the boundary of P at an arbitrary pre-specified point.

Our existence proofs are, in fact, most simply formulated for *continuous* equipartitioning problems (where R and B are replaced by two general mass distributions (measures) on P. In the discrete case, using a natural relaxation of the notion of "equitable", our constructions extend in a direct way to the case when $|R|$ and $|B|$ are not necessarily integral multiples of k.

In section 2 we prove Theorem 1. In section 3 we present an efficient algorithm for computing the partition that is guaranteed by Theorem 1.

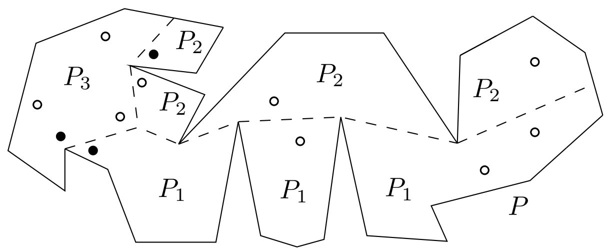


Fig. 1. An equitable 3-partition of a polygon.

2 Existence of convex equipartitions of polygons

As in Theorem 1, we assume that R and B are point sets in a simple polygon P colored in red and blue, respectively, and $|R| = kn$, $|B| = km$.

2.1 2-cuttings

A *2-cutting* is a partition of P into two relatively convex regions (half-polygons) P_1 and P_2 by the geodesic between two points on the boundary of P. A region is called a k_1-*region* if it contains exactly $k_1 n$ red points for some integer $0 < k_1 < k$. A 2-cutting is *equitable* (or more specifically (k_1, k_2)-equitable) if each region P_i contains exactly $k_i n$ red and $k_i m$ blue points, for some integers $k_1, k_2 < k$ with $k_1 + k_2 = k$.

In general we seek regions of P with proportionate representations of red and blue points. Disproportionate regions either have a blue excess (the proportion of blue to red points exceeds m/n) or an excess of red points (the proportion of blue to red points is less than m/n). For simplicity we say the *sign* of a region R

is $+1$ (respectively, -1, 0) if the ratio of blue to red points in R is $>$ (respectively, $<, =$) m/n.

If there is a (k_1, k_2)-equitable 2-cutting, for some integers $k_1, k_2 < k$ with $k_1 + k_2 = k$, then the convex equipartition problem can be directly reduced to the problems of finding relatively convex equitable k_i-partitions of P_i, for $i = 1, 2$. Alternatively, every k_1-region, for $k_1 \in \{1, \ldots, k-1\}$, has a non-zero sign. In fact, for every *fixed* $k_1 \in \{1, \ldots, k-1\}$, every k_1-region must have the *same* non-zero sign:

Lemma 1. *If, for some $k_1 \in \{1, \ldots, k-1\}$, P admits two k_1-partitions with opposite signs, then P has a $(k_1, k-k_1)$-equitable 2-cutting.*

Proof. Essentially, the proof is by continuity and analogous to Lemma 3 in [2] and the proof of the existence of geodesic ham-sandwich cut[3].

Thus we proceed under the assumption that the sign of every k_1-region of P is fixed (and, necessarily, the sign of k_1-regions is opposite that of $k-k_1$-regions). Let p_0 be any fixed point on the boundary of P and let $p_1, \ldots, p_{k-1}$ be points on the boundary of P such that the geodesics p_0p_i, $i = 1, \ldots, k-1$ split P into k regions with n red points in each, see Fig. 2. From our consistency assumption, we define the *sign* of k', $s(k')$, as the sign of the particular k'-region formed by the geodesic $p_0p_{k'}$.

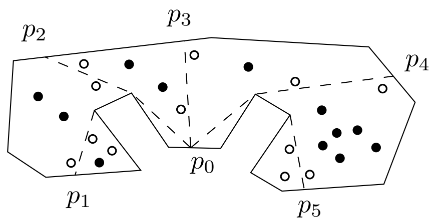


Fig. 2. Partition of P into $k = 6$ regions. The signs are $s(1) = s(2) = s(3) = s(4) = -1, s(5) = 1$. Red points are marked as circles and blue points are solid.

2.2 3-cuttings

A *3-cutting* is a partition of P by three geodesics pa, pb, pc where p is a point of P called the *apex* of the 3-cutting and a, b, c are three points on the boundary of P. A 3-cutting is *relatively convex* if each region of the partition is relatively convex. We will find a 3-cutting with one point from $\{a, b, c\}$ being fixed, specifically our chosen boundary point p_0. We refer to p_0 as the *base point* of our 3-cuttings. (It plays the role similar to a point with $y = -\infty$ in the unconstrained version of the problem [2]). A 3-cutting of P into P_1, P_2, P_3 forms a k_1, k_2, k_3-partition if the region $P_i, i = 1, 2, 3$ contains exactly k_in red points for some integers

$0 \le k_1, k_2, k_3 < g$ with $k_1 + k_2 + k_3 = k$. A k_1, k_2, k_3-partition is *equitable* (or more specifically (k_1, k_2, k_3)-equitable) if the region $P_i, i = 1, 2, 3$ also contains exactly $k_i m$ blue points (that is, each P_i has sign zero).

Our goal is to find an equitable 3-cutting. It is not immediately obvious, what triples (k_1, k_2, k_3) to choose. It turns out that it suffices to choose a triple all of whose signs agree.

Theorem 2. *Let R be any set of kn red points and B be any set km blue points in a simple polygon P in general position. Let k_1, k_2 and k_3 be positive integers with $k_1 + k_2 + k_3 = k$. If $s(k_1) = s(k_2) = s(k_3)$ then there exists either*
(i) *a (k_1, k_2, k_3)-equitable 3-cutting, or*
(ii) *a $(k_i, k - k_i)$-equitable 2-cutting for some i.*

Given Lemma 1 and the following technical lemma concerning sign sequences, our central result (Theorem 1) follows directly from Theorem 2.

Lemma 2. [2, Theorem 9] *For any sequence of non-zero signs $s(1), \ldots, s(k-1)$, where $s(i) \neq s(k-i)$, there is triple (k_1, k_2, k_3) with each $s(k_i)$ having the same sign, $k_1 + k_2 + k_3 = k$ and all $k_i \le 2k/3$.*

As in [2] we prove Theorem 2 by describing a search process that either discovers a (k_1, k_2, k_3)-equitable 3-cutting or terminates early with a $(k_i, k - k_i)$-equitable 2-cutting.

2.3 Canonical 3-cuttings and Geodesic Shield

Similar to [2] we define a *canonical 3-cutting*. Let x be a point in the polygon. We draw geodesic xp_0 and geodesics connecting x and all red points. Since geodesics from x do not cross, these geodesics serve to order the red points about x. Let $r_1, r_2, \ldots, r_{kn}$ be a list of the red points in clockwise order starting from (but omitting) the downward geodesic xp_0. We extend the last edges of geodesics xr_{k_1} and $xr_{k_1+k_2}$ and obtain the *canonical 3-cutting* with three geodesic arms - *left, right* and *downward* geodesics, see Fig. 3. The point x is called an *apex* of the 3-cutting. The geodesic arms split the polygon into "wedges" - *left, right* and *top*. We define $s_i(x), i = 1, 2, 3$ to be the signs of the left, top and right wedges of the canonical 3-cutting with apex x, respectively.

The first part of the proof of the existence of an equitable 3-cutting [2] is the construction of a shield region (the locus of all 3-cutting apexes whose associated canonical wedges are all convex). We show that a "geodesic" shield S can constructed as well. The left side of the shield is the geodesic $p_0p_{k_1}$, see Fig. 4. The right side of the shield is the geodesic $p_0p_{k_1+k_2}$. The top side is defined using geodesics p_0p where p is a point of the polygonal path $p_{k_1}p_{k_1+k_2}$ (in clockwise order). For any point $x \in p_0p$, let $C(x)$ denote the canonical 3-cutting with apex x. Let $\alpha_1, \alpha_2, \alpha_3$ denote the angles formed by the edges incident to p, see Fig. 3. The canonical 3-cutting $C(x)$ has the following property.

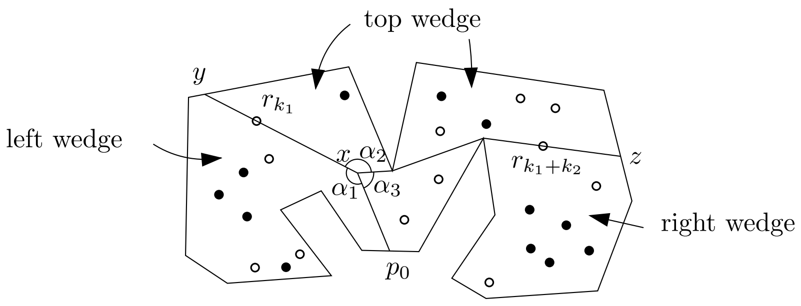


Fig. 3. The canonical 3-cutting with apex x.

Monotonicity. The angle α_2 is non-decreasing when x moves along the geodesic p_0p from p_0 to p. The angles α_1 and α_3 are non-increasing when $\alpha_2 > 0$ and x moves along the geodesic p_0p from p_0 to p.

One can show that if x moves far enough from p_0 than α_2 becomes greater than or equal to π. More specifically, suppose that x crosses the geodesic $p_{k_1}p_{k_1+k_2}$ and let xy and xz be the left and right geodesics of canonical 3-cutting with apex x, see Fig. 4. Then y lies on the polygonal path $p_0p_{k_1}$ and z lies on the polygonal path $p_{k_1+k_2}p_{k_1}$. Thus angle $\alpha_2 > \pi$. In general the stop condition for the motion of x is that two arms, left and right, form one geodesic yz. If x moves beyond this point then α_2 becomes greater than π. Prior to the stop condition the canonical 3-cutting with apex x defines a relatively-convex partition. The locus of all such points is called a *(geodesic) shield* with base point p_0, see Fig. 4.

It turns out that properties of the shield boundary alone allow us to conclude that it must contain a the apex of an equitable 3-cutting.

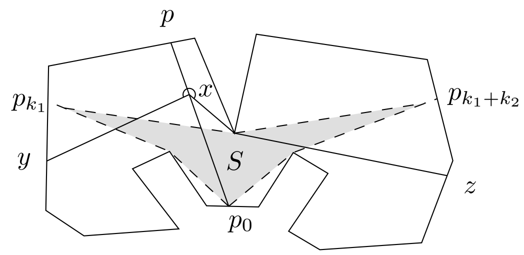


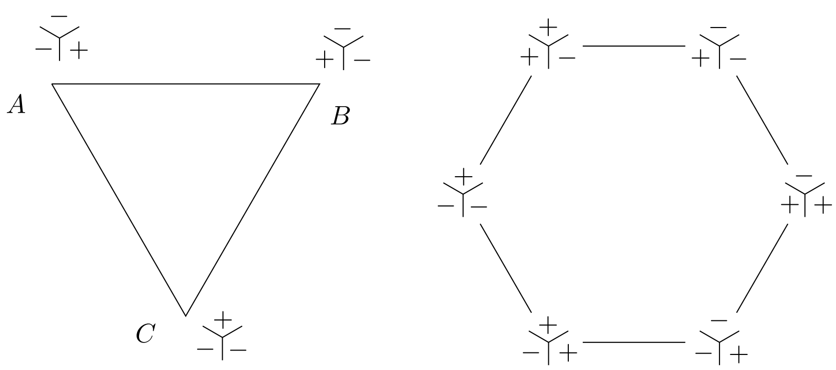
Fig. 4. The shield S defined by (k_1, k_2, k_3) is shaded.

2.4 Existence of equitable 3-cutting

At this point our proof, which has closely paralleled that of [2], switches to exploit the tools developed in [1] for proving the existence of constrained 3-cuttings.

Since every point x in the shield S has three associated wedges it has three associated signs. We define s_i*-levels* as the sets of points $x \in S$ where $s_i(x) = 0$. The shield S can be mapped into a regular triangle ABC where A, B and C are the images of $p_{k_1}, p_{k_1+k_2}$ and p_0, respectively. By the construction of the shield, the signs of A, B and C are as depicted in Fig. 5 (a). The levels have a property that

1. the geodesic corresponding to AC does not contain an endpoint of a s_1-level, and
2. the geodesic corresponding to AB does not contain an endpoint of a s_2-level, and
3. the geodesic corresponding to BC does not contain an endpoint of a s_3-level.



(a) (b)

Fig. 5. (a) Triangle ABC. (b) Label diagram.

It follows that a clockwise walk around the boundary of the shield encounters apexes whose sign triples trace exactly one (counterclockwise) turn of the label graph in Fig. 5 (b).

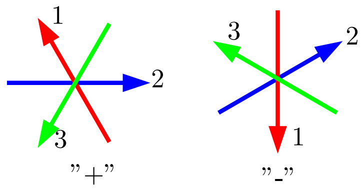


Fig. 6. Orientation of intersecting levels.

The arrangement of levels on the shield has many interesting properties. Obviously, if two levels intersect at a point all three must intersect. If we orient levels such that the left side of an oriented level has negative sign s_i corresponding to the level, the levels intersecting in points with *alternating directions* in-out-in-out-in-out, see Fig. 6. We label a vertex as "+" if the outcoming levels form the sequence 123 in clockwise order, see Fig. 6 c). Otherwise it has label "-". Let v^+, v^- denote the number of the internal vertices labeled by "+" and "-" respectively. Let b^+, b^- denote the number of the external edges of the arrangement adjacent to internal vertices with corresponding labels.

For our purposes, the main property of these level arrangements is the following.

Theorem 3. *Let T be any connected region in P and suppose that T does not have a level intersection point on its boundary. Let t be the number of (clockwise) turns of the closed path on the label graph 5 (b) when one walks on the boundary of T in clockwise order. Then*

$$v^+ - v^- = \frac{b^+ - b^-}{6} = t. \tag{1}$$

If T is the triangle ABC, then $t \equiv -1 \mathtt{mod} 3$.

It follows immediately from Theorem 3 that v^+ and v^- cannot both be zero, that is there must be at least one internal intersection point in the arrangement of levels.

3 Algorithmic results

The algorithm overview is as follows.

Geodesic-Partition(P, R, B)

1. Construct $k-1$ geodesics that partition P into k regions with n points in each. Compute the signs $s(i), i = 1, \ldots, k-1$.
2. By Theorem 7 [2] there exists either a pair $(a, b), s(a) = s(b), a + b = k$ or a triple $(a, b, c), s(a) = s(b) = s(c), a + b + c = k$. Such a pair or a triple can be computed in $O(k)$ time.
3. If a pair (a, b) is found then compute (a, b)-equitable 2-cutting.
4. If a triple (a, b, c) is found then apply the binary search with geodesics p_0p and compute a funnel bounded by two geodesics p_0p and p_0q. Find an apex of equitable (a, b, c)-cutting in the funnel p_0pq.

Step 3 can be done using the technique by Bose *et al.* [3] which can be viewed as a special case $a = 1/2$. This takes $O(N \log N)$ time (the factor of $\log N$ can be even improved to $\log n_r$ where n_r the number of reflex vertices of P). We focus on Step 4 and show that it can be performed in $O(N^2 \log N)$ time.

Let $\mathcal{A}$ be an arrangement of all straight line segments s such that (i) the endpoints of s are on the boundary of P, and (ii) s contains two points from

$R \cup B \cup V$ where V is the set the vertices of the polygon P. The arrangement $\mathcal{A}$ has a property that, for any two points a and b from a face $f \in \mathcal{A}$ or an edge $e \in \mathcal{A}$, the signs are equal $s_i(a) = s_i(b), i = 1, 2, 3$. We can assign the signs $s_i(a), i = 1, 2, 3$ to the face f or the edge e. By Theorem 3 there exists either a vertex or an edge or a face of $\mathcal{A}$ with the signs $(s_1, s_2, s_3) = (0, 0, 0)$. One way to find such an element of $\mathcal{A}$ is to compute the signs for all elements of $\mathcal{A}$. This leads to an $\Omega(N^4)$ algorithm since the complexity of $\mathcal{A}$ is $\Theta(N^4)$ in the worst case. Our goal is to design a more efficient algorithm.

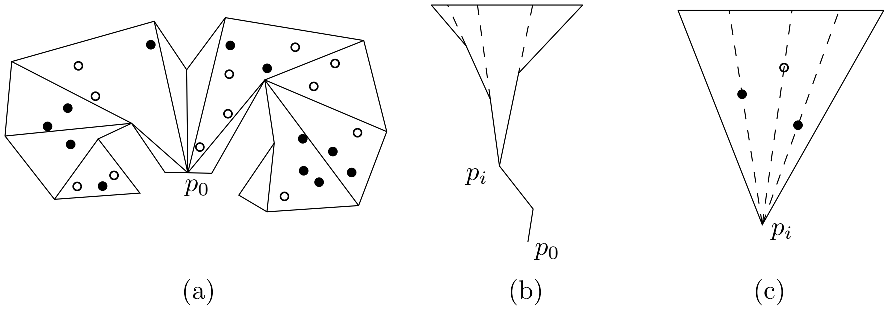


Fig. 7. Binary search: (a) shortest paths tree, (b) triangulation of a funnel, (c) sorting colored points.

We extend the idea [2] of applying a binary search to avoid checking all elements of $\mathcal{A}$. However the problem is more complicated since the polygon P restricts the search region. First, we construct the shortest paths tree from p_0, see Fig. 7 (a). The shortest paths tree splits the polygon into funnels and we partition the funnels into triangles, see Fig. 7 (b). We locate the colored points in the triangulation and sort them, see Fig. 7 (c). Thus the points of $R \cup B \cup V$ are sorted by geodesics rooted at p_0.

We apply binary search using the sorted order of $R \cup B \cup V$. Let p_0p be a geodesic that splits the polygon P into two polygons P_1 and P_2. We want to decide whether P_i contains the apex of an equitable 3-cutting. By Theorem 3 the decision can be made if we know the path $\Pi(p_0p)$ on the label diagram 5 (b) when one walks along the geodesic p_0p. The geodesic p_0p crosses $O(N^2)$ faces of the arrangement $\mathcal{A}$. It suffices to trace only the signs s_1 and s_3 since $s_2 = -s_1 - s_3$ for any point from the geodesic p_0p. The computation of signs s_1 (resp. s_3) can be done using the polygon P_1 (resp. P_2). By symmetry it suffices to show how to compute the signs s_1 along the geodesic p_0p.

For every vertex r of P_1 we construct the shortest path tree for it with sorting points of $R \cup B$ in the triangles as we did for p_0. This gives the sorted order of $R \cup B \cup V$ about r. Let $G_r = \{rr'\}$ be the set of these geodesics and let B_r be *base* points r' of the geodesics rr'. We associate a base point with an edge of P_1 that contains it. By doing this we collect a set of base points B_e for each edge of P_1. There are two types of base points, see Fig. 8.

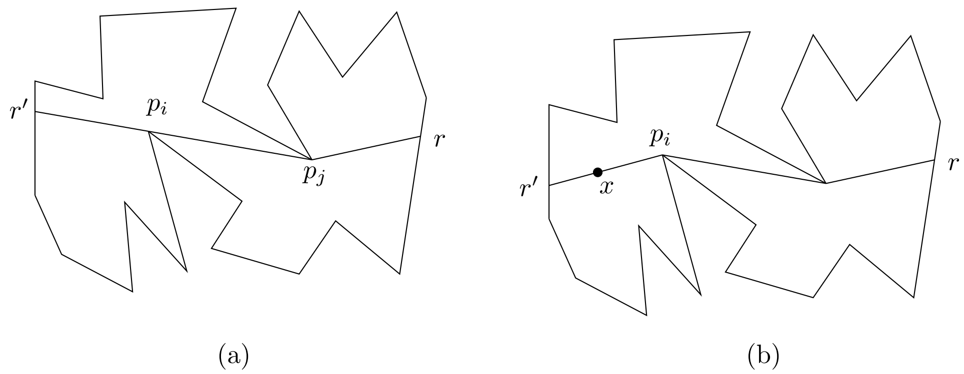


Fig. 8. Two types of the base points. (a) r' lies on the line $p_i p_j$. (b) r' lies on the line $p_i x$ where x is a colored point.

To compute the sequence of signs s_1 along the path $p_0 p$ we move a point r from p_0 to p and maintain the geodesic rr' that is the left geodesic of the canonical 3-cutting in P. There are two types of events when r moves:

1. r meets a vertex of P_1. We change the edge that r traverses.
2. r lies on the line l passing through p_i and p_j for two vertices of $p_i, p_j \in P_1$. Wlog p_j is in between p_i and r. If $p_i p_j$ is the edge of the geodesic rr' then p_j drops from rr'. Otherwise we insert p_j into rr'.
3. r lies on the line l passing through $p_i \in P_1$ and a colored point $x \in R \cup B$. If x is blue we change the counting for s_1. If x is red then we move r' counterclockwise until it hits a red point. Then we continue to move r. We maintain the counting for s_1 when r' moves.

Theorem 4. *Let $k \geq 2$ and let R and S be point sets, consisting of kn red points and km blue points respectively, in general position in the interior of a simple polygonal region P. An equitable 2-cutting or 3-cutting can be computed in $O(N^2 \log N)$ time where $N = k(n+m) + |P|$.*

By applying the above algorithm recursively we obtain

Corollary 1. *Let $k \geq 2$ and let R and S be point sets, consisting of kn red points and km blue points respectively, in general position in the interior of a simple polygonal region P. There exists an equitable relatively-convex partition of P into k polygons and it can be computed in $O(N^2 \log N \log k)$ time.*

4 Conclusion

We proved the existence of an equitable relatively-convex partition of a simple polygon with red and blue points inside. We developed an efficient algorithm for computing the partition. Improving the running time of the algorithm is a direction for further research.

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