All Convex Polyhedra can be Clamped with Parallel Jaw Grippers

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Abstract

We study various classes of polyhedra that can be clamped using parallel jaw grippers. We show that all n-vertex convex polyhedra can be clamped regardless of the gripper size and present an $O(n+k)$ time algorithm to compute all positions of a polyhedron that allow a valid clamp where k is the number of antipodal pairs of features. We also observe that all terrain polyhedra and orthogonal polyhedra can be clamped and a valid clamp can be found in linear time. Finally we show that all polyhedra can be clamped with some size of gripper.

1 Introduction

Grasping is an active research area in robotics. Much research has been done on the problem of gripping or immobilizing an object with a multi-fingered hand [4, 8, 12, 13, 16]. Motivated by robot hands consisting of pairs of parallel rectangular plates (known as *parallel jaw grippers*) researchers have also considered the problem of finding a secure grip of a planar object with a pair of parallel line segments [6, 14, 17, 19]. Each plate is referred to

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as a *gripper*. Informally, a polygon P is *clamped* in the plane when it is "securely" held between the two grippers (modeled in the plane by a pair of line segments forming the opposite sides of a rectangle) such that P does not rotate or slip out of the gripper when the gripper is squeezed. A polygon is called *clampable* if there exists a clamp for every positive length gripper.

Souvaine and Van Wyk [17] showed that all convex polygons are clampable, and conjectured that all simple polygons are clampable. Capoyleas [6] gave a slightly weaker definition of clamping and showed that a class of polygons with convex pockets is clampable under this definition. Albertson, Haas, and O'Rourke [2] defined the class of *free* polygons (a polygon is called free if no outward normal of the polygon intersects its interior) and showed that free polygons are clampable. They also showed that sail polygons (a polygon is a sail polygon if it has exactly three convex vertices), and polygons with at most 5 vertices are clampable.

In this paper we address the problem of determining when a 3-dimensional object (modeled as a simple polyhedron) is clampable with a parallel jaw gripper consisting of a pair of parallel plates that are the opposite faces of a rectangular box. First, we observe that terrain polyhedra and orthogonal polyhedra are clampable and that a valid clamp can be found in linear time. The main result of our paper is a proof that all convex polyhedra are clampable. We also give an $O(n + k)$ time algorithm to find all clamps of an n-vertex convex polyhedron, where k is the number of antipodal pairs of features of the polyhedron.

2 Notation and Preliminaries

All polygons and polyhedra considered in this paper will be simple; for definitions of this and other geometric terms the reader is referred to e.g. [14, 15]. We will denote the open interior of a polyhedron P by $int(P)$. Polyhedra are closed, i.e. the boundary is considered part of the polyhedron. The convex hull of a set S of points is denoted $CH(S)$. The closed line segment with endpoints x and y is written $\overline{xy}$. The relative interior of a point set S is the interior of S in the highest dimensional affine subspace spanned by S; for a line segment $\overline{ab}$ this is $\overline{ab} \setminus \{ a, b \}$ and for a point p this is p itself. Two point sets *intersect properly* if their relative interiors intersect.

2.1 Grippers and Clamps

A parallel jaw gripper is modeled by a pair of rectangles forming opposite faces of a rectangular box. Each rectangle is referred to as a gripper. The size of a gripper is determined by the length and the width of the rectangle. Although intuitively it may seem that the size of a gripper plays a key role in determining whether or not a polyhedron can be clamped, we will show that several classes of polyhedra can be clamped regardless of the size of the gripper.

To give a geometric definition of a clamp, we begin by first defining the *contact set* of a gripper (similarly to [17]). We shall give a characterization of the secureness of a configuration of the grippers and the object that is based only on the points of the grippers in contact with the object. Let $\mathrm{dist}(p, q)$ denote the Euclidean distance between p and q. Given a polyhedron P and a gripper G, the *contact set* of G is the set of all points $p \in G$ such that for all $\epsilon > 0$, there is a point q that lies in both the interior of the box defined by the grippers and in the interior of P such that $\mathrm{dist}(p, q) < \epsilon$.

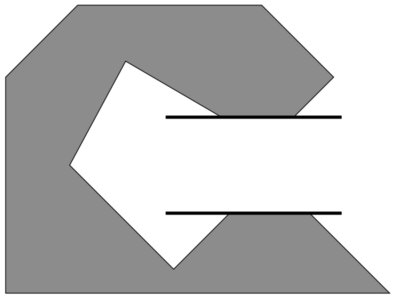

Figure 1: An empty contact set

Note that the contact set may be a proper subset of the set of all points at which G touches polyhedron P. The ϵ-condition for the contact set ensures that the polyhedron is between the two grippers and avoids configurations such as the one illustrated in Figure 1. If in some configuration both grippers have non-empty contact sets, where the contact sets are denoted by α and β, we call the pair (α, β) of contact sets a *grip*.

Although when a polyhedron is in a grip both grippers touch the polyhedron and at least some part of the interior of the polyhedron is contained between them, the grip is not necessarily secure. By this we mean that by squeezing the grippers, the polyhedron may rotate or slip out (see e.g. Figures 2 and 3). Therefore, we wish to define a grip that “holds a polyhedron securely”. Such a grip will be referred to as a *clamp*.

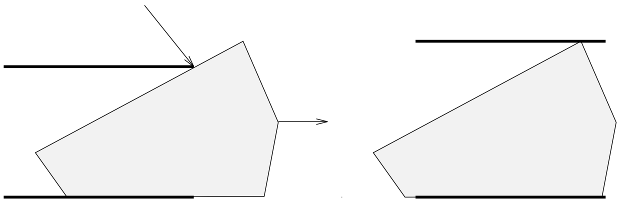

Figure 2: A grip where the object translates when the grippers are squeezed, compared with a secure alternative.

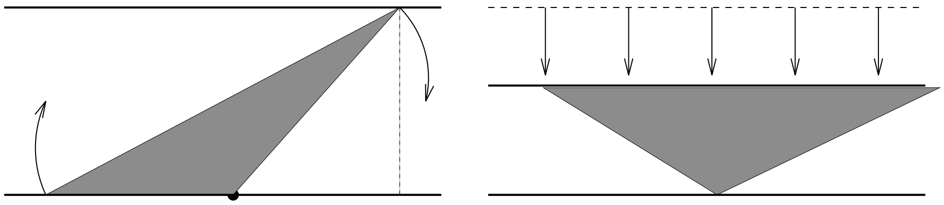

Figure 3: A grip where the object rotates when the grippers are squeezed, along with the secure grip reached after rotation.

In the rest of this section, we consider a simplified physical model of a rigid object gripped by parallel jaw grippers, and use this model to define a clamp geometrically. For a more rigorous treatment of the statics of rigid bodies, we refer the reader to [18]. We start from the premise that a grip is secure if and only if the object does not translate or rotate (relative to the grippers) when the grippers are squeezed. This is equivalent to requiring that the system of the grippers and the object be in *static equilibrium*, i.e. the external forces and torques acting on the object sum to zero. In general there will be other forces acting on an object other than those exerted by the grippers; we assume that when the grippers are able to exert arbitrarily large pressure on the object without dislodging it, the friction between the grippers and the object (which is proportional to the pressure of the grippers) counterbalances any external forces such as gravity.

In the following, we assume without loss of generality that the grippers are horizontal and the z-axis is normal to the grippers. The x and y axes will thus be contained in a plane parallel to the grippers. Let $\mathrm{proj}_\Gamma(A)$ denote the orthogonal projection of A into plane Γ. Let $\mathrm{proj}(A)$ denote the orthogonal projection of A into the plane $z = 0$. For ease of reference, we define the following condition on a pair of contact sets (α, β), illustrated in Figure 4.

Condition 1

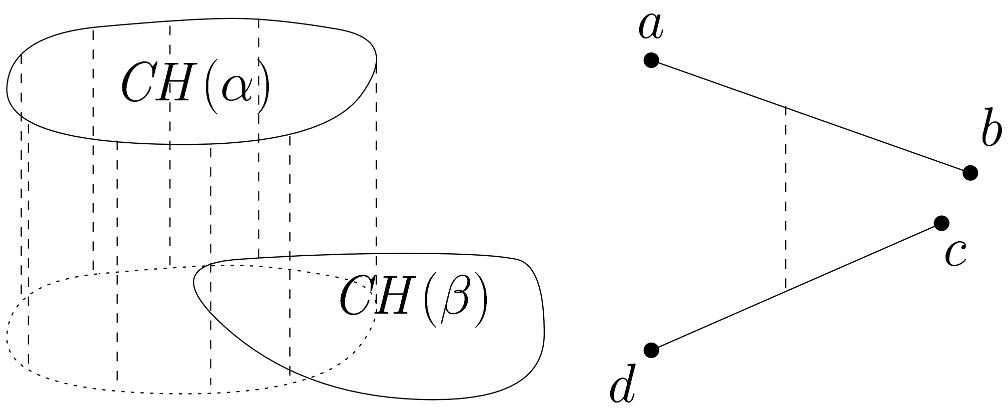

Figure 4: Illustrating the two cases of Condition 1.

(a) One of $CH(\alpha)$ *or* $CH(\beta)$ *has non-zero area and* $\mathrm{proj}(CH(\alpha))$ *properly intersects* $\mathrm{proj}(CH(\beta))$*,*

or

(b) There exists $(a, b) \in \alpha$ *and* $(c, d) \in \beta$ *such that* $\mathrm{proj}(\overline{ab})$ *properly intersects* $\mathrm{proj}(\overline{cd})$.

We would like our definition of a clamp to exclude cases such as the one illustrated in Figure 5, since they are liable to be insecure if they are misaligned slightly. We say that an object is in *stable rotational equilibrium* if the sum of the torques on it is zero and there exists some $\epsilon > 0$ such that if each coordinate of the contact points is perturbed by any δ such that $|\delta| < \epsilon$, the sum of the torques is still zero. Stable static equilibrium is defined analogously.

Lemma 1 *If an object held in a grip* (α, β) *is in stable rotational equilibrium with respect to any axis normal to the* z*-axis then Condition 1 holds for* α *and* β*.*

Proof: Suppose the object is in stable rotational equilibrium about every axis of rotation perpendicular to the z-axis. It follows $\mathrm{proj}(CH(\alpha))$ must properly intersect $\mathrm{proj}(CH(\beta))$, hence if either $CH(\alpha)$ or $CH(\beta)$ has non-zero area, then Condition 1a holds.

Suppose $CH(\alpha)$ and $CH(\beta)$ both have zero area. If α and β are coplanar then the rotational equilibrium is not stable, hence Condition 1b holds. ∎

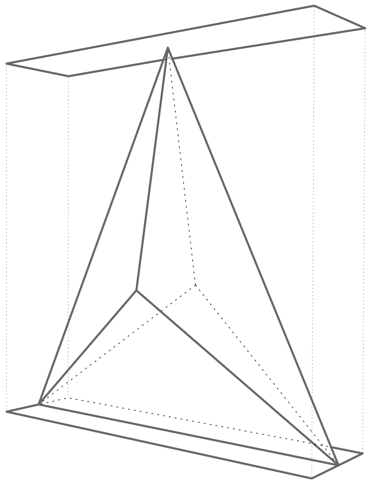

Figure 5: A grip in static equilibrium that is unstable

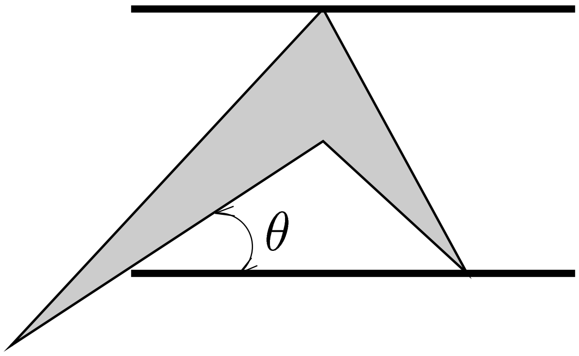

Figure 6: A grip whose stability depends on the angle θ.

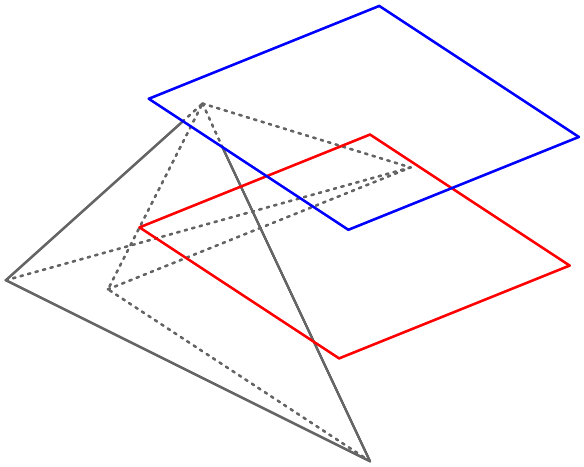

Figure 7: An unstable grip where Condition 1a is satisfied (the bottom gripper is above and parallel to the base of the object).

We have established that stable static equilibrium implies Condition 1 (since static equilibrium implies rotational equilibrium). The converse is unfortunately not true. Even in two dimensions, if one of the contact sets contains points on the boundary of a gripper, then the the object may translate or rotate when the grippers are squeezed even though (the two dimensional equivalent of) Condition 1 is satisfied. Consider for example the grip shown in Figure 6; the stability of this grip seems to depend not only on the contact sets, but on the angle θ. Even stronger examples can be constructed in three dimensions (see e.g. Figures 7, 8 and 9).

In order to have static equilibrium, not only must Condition 1 hold, but also the force components acting along the x and y axes and the torque components about the z axis must sum to zero. One trivial way of ensuring this is to insist that the object does not contact the edge of the grippers; in this case all forces have vertical lines of action. We give a weaker sufficient condition, although it is still not necessary. Given a grip (α, β), let the *proper contact sets* (α', β') denote those points in the contact sets contained in the open interior of the respective grippers.

Lemma 2 *If the proper contact sets* (α, β) *satisfy Condition 1, then the object is in stable rotational equilibrium w.r.t. any axis normal to the* z*-axis.*

Proof: Suppose that Condition 1 holds for proper contact sets α and β. Let Γ be an arbitrary vertical plane. One of $A \equiv \mathrm{proj}_\Gamma(CH(\alpha))$ and

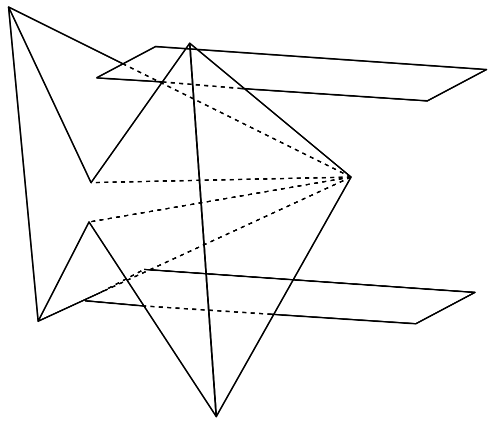

Figure 8: An unstable grip where Condition 1b is satisfied.

$B \equiv \mathrm{proj}_\Gamma(CH(\beta))$ must be a line segment. Without loss of generality, suppose B is a line segment. Since there is a point in $CH(\alpha)$ that projects orthogonally into the open relative interior of $CH(\beta)$, the line segments $\mathrm{proj}(A)$ and $\mathrm{proj}(B)$ must overlap properly. It follows that one of them, without loss of generality $\mathrm{proj}(A)$, must have an endpoint that projects in the interior of the other, unless both line segments are equal. But the endpoints of $\mathrm{proj}(A)$ and $\mathrm{proj}(B)$ are projections of vertices of $CH(\alpha)$ and $CH(\beta)$, hence points in α and β respectively. Observe that in each case, there is a configuration of three or four points in the projection onto Γ corresponding to what Albertson, Haas, and O'Rourke [2] call a three or four point clamp in two dimensions. Given one of these configurations of contact points in the interior of the grippers, if the object were to rotate about an axis normal to Γ, it would force the grippers farther apart, a contradiction. From the definition of rotational equilibrium, it follows that the sum of all torque components in the plane Γ must be zero. Since Γ was an arbitrary vertical plane, the sum of torque about any horizontal axis must be zero.

Note that because Condition 1 requires proper intersection of the projected contact sets, there is always some ϵ perturbation that can be applied to the coordinates of the contact points without violating Condition 1, hence the rotational equilibrium is stable. ∎

Theorem 1 *If Condition 1 holds for proper contact sets α and β, the object is in stable static equilibrium.*

Proof: Suppose Condition 1 holds and there exists points $a \in \alpha$ and $b \in \beta$

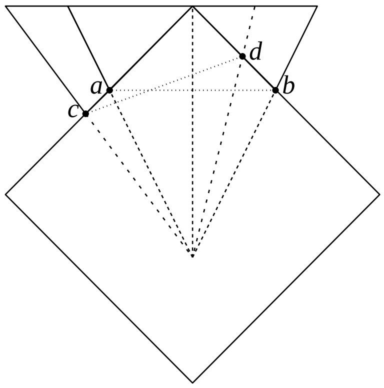

Figure 9: Projection of Figure 8 into the plane $z = 0$.

in the interior of the grippers. In order for there to be translation of the object or rotation about the z axis, the interior of some face of the object must contact the edge of the grippers, since otherwise all forces act parallel to the z-axis. In order for the object to rotate or translate the grippers must come closer together when squeezed. Consider the line segment $\overline{ab}$. Since both a and b are interior to the grippers, if the grippers are moved closer together, $\overline{ab}$ must rotate about some horizontal axis. Because the object is rigid, this would imply that the whole object rotates about a horizontal axis. But by Lemma 2, we know this is impossible. ∎

This leads us to define a clamp as follows:

Definition 1 *A grip is a clamp if Condition 1 holds for the proper contact sets.*

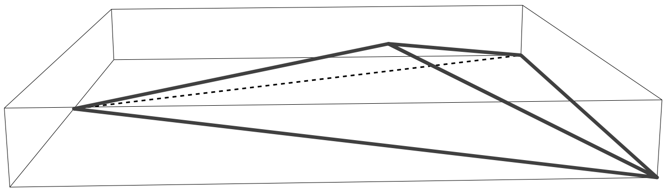

Figure 10: Only a vertex-face clamp exists.

The tetrahedron in Figure 10 can be clamped under condition 1a with one gripper on the peak of the tetrahedron and the other on its base. The

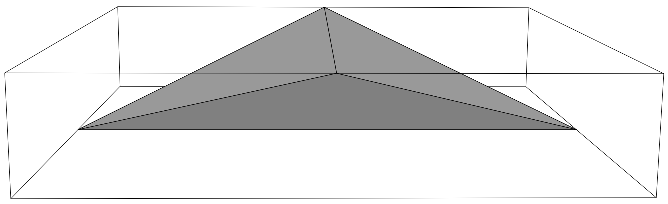

Figure 11: Only an edge-edge clamp exists.

tetrahedron in Figure 11 can be clamped under condition 1b. Figure 12 illustrates that there are stable grips that do not satisfy Definition 1. Such grips require a more sophisticated characterization but are not needed for the classes of polyhedra studied in this paper.

We say that a polyhedron is *clampable* if it admits a clamp for every size of gripper. We say that a polyhedron is *partially clampable* if it admits a clamp with a gripper of a particular size.

3 Clampable Polyhedra

The definition of a clamp immediately implies that all terrain polyhedra are clampable. A polyhedron P is a terrain polyhedron provided that there exists a face f of P contained in a plane F such that $\forall x \in P, \overline{xy} \subset P$, where $y = \mathrm{proj}_F(x)$. Such polyhedra can be recognized in linear time [3].

Observation 1 *Every terrain polyhedron is clampable and a clamp can be found in linear time.*

Every orthogonal polygon has at least 4 extreme edges (edges incident on the minimum enclosing rectangle). Since the two edges adjacent to an extreme edge are parallel and project orthogonally onto each other for some positive distance, there exist at least 2 distinct clamps of a orthogonal polygon. Similarly, every orthogonal polyhedron has at least 6 extreme faces. By considering the set of extreme faces in a given direction, finding a clamp of a orthogonal polyhedron reduces to finding an edge that is extreme for a set of orthogonal polygons.

Observation 2 *Every orthogonal polyhedron is clampable and a clamp can be found in linear time.*

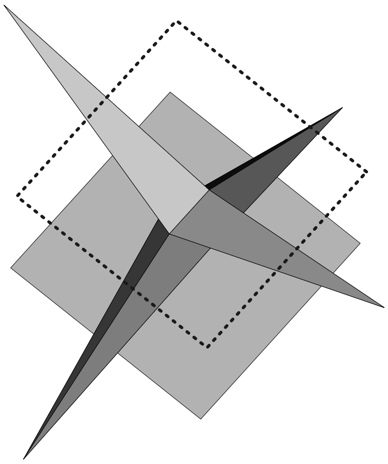

Figure 12: A stable grip that does not satisfy Definition 1. The vertices form two nested regular tetrahedra.

3.1 Convex polyhedra

In this subsection, we establish our main result: that all convex polyhedra are clampable. We also give an algorithm to determine all positions that admit a valid clamp.

To show that all convex polyhedra may be clamped, we must show that given a gripper of any size, there always exists at least one position of the grippers that satisfies one of the two conditions defining a clamp. A key towards showing this is the observation that a convex polyhedron can only be clamped at an *antipodal* pair of features. An antipodal pair of features is the intersection of a convex polyhedron with a pair of parallel support planes. Since a plane of support can only meet a convex polyhedron at a vertex, edge or face, there can only be six types of antipodal pairs: vertex-vertex, vertex-edge, vertex-face, edge-edge, edge-face and face-face.

From the definition of a clamp, we see that a vertex-vertex pair and vertex-edge pair cannot form a clamp. Therefore, what remains to be shown is that there always exists an antipodal pair satisfying one of the two criteria of clamping. There are two special types of antipodal pairs that immediately come to mind: those that determine the maximum (*diameter*) and minimum (*width*) distances between parallel planes of support. The diameter can be determined by an antipodal pair that does not satisfy Condition 1. In [11], the authors show that the width cannot be determined by a vertex-vertex or vertex-edge pair that is not part of another antipodal pair.

Lemma 3 [11] *The width of a convex polyhedron P in three dimensions is determined by an antipodal vertex-face pair, edge-edge pair, face-face pair, or edge-face pair of P.*

Lemma 3 in itself is not sufficient to show that all convex polyhedra are clampable. We now show, however, that an antipodal pair determining the width always satisfies one of the two conditions defining a clamp. Before proving this theorem, we need to establish a few geometric lemmas. First, we show that for convex polyhedra, there is no need to distinguish between contact sets and *proper* contact sets in the interior of the grippers. Since P is convex, then so are the contact sets. It follows that if we remove points on the boundary of the contact sets (and the grippers), Condition 1 will still be satisfied. Thus we have the following:

Lemma 4 *For convex polyhedra, if Condition 1 is satisfied for a grip G, then G is a clamp.*

The following lemma is a specialization of (1.10) in [9] to 3-dimensional space and the Euclidean metric.

Lemma 5 [9] *Let P be convex polytope. If A and B are parallel planes of support for P realizing the width of P, then $\mathrm{proj}_B(A \cap P)$ properly intersects $B \cap P$.*

In essence, the above lemmas tell us that convex polyhedra are clampable with infinite grippers. We now show that the grippers can in fact be made arbitrarily small.

Lemma 6 *If there is a clamp of a convex polyhedron P for some positive size gripper, there is a clamp for any positive size gripper.*

Proof: To see this is true, we note that in both cases of Definition 1 there is some point p in the proper intersection of the two projected contact sets. Since P is convex, the points in the convex hulls of two contact sets that project onto p must be points on the surface of P. It follows that that if we keep the grippers centered on p then Condition 1 will continue to hold no matter how far we shrink the grippers (while maintaining a positive size). ■

We now have the tools to prove the following theorem, a generalization of the main result of [17].

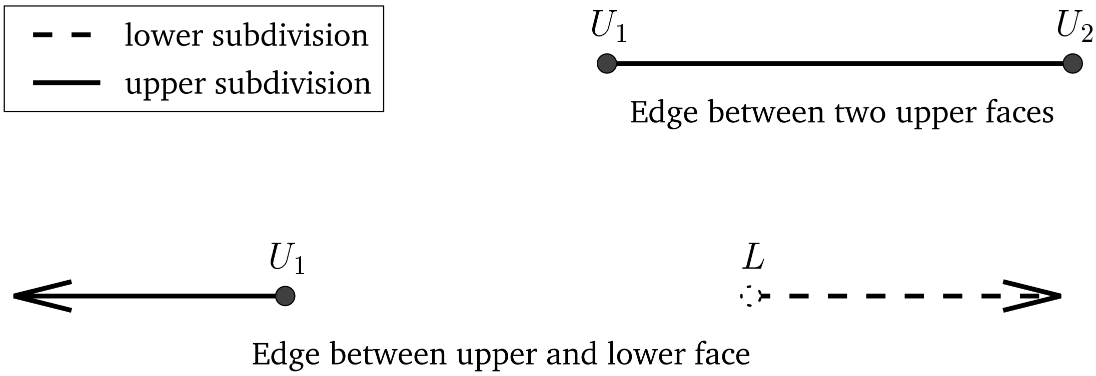

Figure 13: The two types of dual edges.

Theorem 2 *An antipodal pair determining the width of a convex polyhedron provides a clamp for any positive size gripper.*

Proof: Let P be a convex polytope. Consider a pair of rectangular grippers placed on an antipodal pair of features (p, q) of P that determine the width. For sufficiently large grippers, the contact sets are precisely (p, q), and the grip is a clamp by Lemmas 3, 4 and 5. But by Lemma 6, it follows that there is a clamp for any positive size gripper. ∎

Corollary 1 *Every simple polyhedron is partially clampable.*

Proof: Notice that a clamp is defined in terms of the convex hulls of the contact sets, rather than the contact sets. The convex hulls of the contact sets are precisely the contact sets of the convex hull, so we may apply Lemmas 3 and 5 to the convex hull of the input polytope. We can always enlarge the grippers so that the contact set is entirely contained in the interior of the grippers. ∎

Although there can be many positions admitting a valid clamp, there are some polyhedra (such as the ones depicted in Figures 10 and 11) where the antipodal pair determining the width is the only position providing a clamp.

We now turn our attention to computing a valid clamp. Theorem 2 guarantees that every convex polyhedron has at least one valid clamp. To find such a clamp, we rely on Brown's technique [5] to compute all antipodal pairs of features. We briefly summarize this technique. Given a convex polyhedron tangent to the $z = 0$ plane with no vertical faces (such an orientation can always be found), the first step is to partition the faces into those whose outward normals have a positive z-component (the *upper* set) and those whose outward normals have a negative z-component (the *lower* set). This division has the property that any antipodal pair of features must have one plane of support tangent to the upper set and one plane of support tangent to the lower set.

A feature U (face, vertex, edge) in the upper set is antipodal to a feature L in the lower set exactly when U and L have supporting planes with the same slope. The upper and lower sets are transformed to upper and lower convex subdivisions, where each feature of a subdivision corresponds to the slopes of planes supporting a given feature of the polyhedron. We describe the computation of the upper subdivision; the lower is symmetric. Each supporting plane $z = ax + by + c$ is mapped to the point $(a, b) \in \mathbb{R}^2$. A face of the upper set maps to a vertex of the upper subdivision since a face has only one plane of support. An edge adjacent to two faces in the upper set is transformed to an edge between the two vertices in the upper subdivision representing the transformed faces. An edge adjacent to a face in the upper set and a face in the lower set is transformed to an infinite ray in the upper subdivision. This ray emanates from the vertex representing the upper face and is directed away from the point representing the lower face (see Figure 13). The vertices of the upper set map to faces in the upper subdivision. The faces of the subdivisions need not be computed explicitly, since once the vertices and edges are computed, the faces are represented implicitly.

Consider the overlay of the upper and lower subdivisions. A vertex f of one subdivision that lies in a face v of the other corresponds to a vertex-face antipodal pair of features because f corresponds to a face and v corresponds to a vertex. The other types of antipodal pairs similarly correspond to features of the overlay (see Figure 14). In particular each of the antipodal pairs that can provide a clamp will be a vertex of the overlay. Therefore to generate all valid clamps, we need only consider the vertices of the overlay. From the overlay we can tell if the antipodal pair is of the appropriate type; by testing directly in the primal space, we can tell if the projection conditions of Definition 1 are met. Theorem 2 guarantees that at least one pair will be valid.

Theorem 3 *Every n-vertex convex polyhedron is clampable and all clamping positions can be computed in $O(n+k)$ time where k is the number of antipodal pairs of features.*

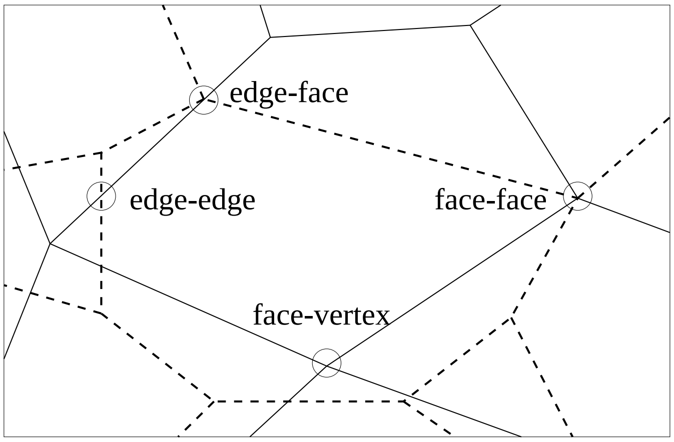

Figure 14: The subdivisions corresponding to the transformed upper and lower sets of facets.

Proof: We have outlined an algorithm whose correctness follows from Theorem 2 and the discussion above. We analyze the complexity of the algorithm below. Throughout this discussion, n is the number of vertices in the polyhedron and k is the number of antipodal pairs of features.

Computing the upper and lower subdivisions can be achieved in $O(n)$ time using the algorithm of Brown [5]. The two subdivisions can be overlayed in $O(n+k)$ using Guibas and Seidel's algorithm [10]. Any edge-edge antipodal pair can be checked for validity in constant time, so total time spent checking edge-edge pairs is $O(k)$. Every other possibly valid antipodal pair contains at least one facet. Checking a face-face antipodal pair amounts to intersecting two convex polygons, which can be done in $O(e)$ time,where e is the number of edges in the two faces [20, 14]. Antipodal pairs involving a face and a vertex or edge (i.e. as part of a second face) can be checked naively in $O(e)$ time, where e is the complexity of the face. We charge the work checking antipodal pairs involving faces to the edges of those faces. Since each face occurs in exactly one antipodal pair, and each edge occurs in exactly two faces, the total amount of work checking this second class of antipodal pairs is $O(n)$ by Euler's formula. ∎

4 Conclusions

We addressed the problem of clamping a three dimensional object using parallel jaw grippers consisting of a pair of parallel rectangular plates. We defined a physical model of what constitutes a *secure* grip by the grippers. We then provided a geometric interpretation of this model and subsequently showed that all convex polyhedra, orthogonal polyhedra and terrain polyhedra are clampable, and all simple polyhedra are partially clampable under this model. We also noted that orthogonal polygons as well as terrain polygons are clampable. We provided a linear time algorithm for determining a clamp on orthogonal polyhedra and terrain polyhedra and provided an $O(n + k)$ time algorithm (where n is the number of vertices and k the number of antipodal pairs of features of the polyhedron) for determining all clamps on a convex polyhedron.

Since there may be $\Omega(n^2)$ antipodal pairs, the algorithm here has worst case performance $O(n^2)$. This is open to improvement in two ways. It would be desirable to have an algorithm sensitive to the total number of valid clamps, rather than the number of antipodal pairs; the existence of such an algorithm is an open problem. On the other hand, if only a single clamp is desired, then it is possible to obtain a better worst case upper bound. Chazelle et al. [7] gave an $O(n^{8/5+\epsilon})$ algorithm for computing the width of a set of points in 3-space. Recent work by Agarwal et al. [1] provides an $O(n^{3/2+\epsilon})$ randomized algorithm for the same problem. Both of these algorithms can be modified to return the antipodal pair defining the width, rather than just the numerical value. They both, however, use sophisticated theoretical techniques whose practicality is unclear. It remains open whether there is a simple $o(n^2)$ algorithm to find a single clamp of a convex polyhedron.

In this paper we restricted ourselves to secure grips generated by contact points in the interior of the grippers. Such grips are not possible for all polyhedra; for sufficiently small grippers, the polyhedron shown in Figure 12 does not admit a clamp satisfying Definition 1 (and yet it does admit a stable grip). On the other hand, allowing clamps that rely on contact with the edge of the grippers requires a more sophisticated approach to analyzing the stability of a grip, as illustrated in Figure 6.

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