Polygon Cutting: Revisited

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Abstract. Given a simple polygon P on n vertices $v_0, v_1, \ldots, v_{n-1}$ with each edge assigned a non-negative weight w_i, we show how to compute in $O(n)$ time a segment v_ib (where b is a point on the boundary) that partitions P into two polygons each having weight at most 2/3 the weight of P. If instead of edge weights, we consider an infinitely additive measure of the interior (such as the area of P), there still exists a segment v_ib that partitions P into two polygons each having measure at most 2/3 the measure of P. In the case where P contains k points in its interior with each point assigned a non-negative weight, then in $O(n + k \log n)$ time a segment v_ib can be computed that partitions P into two polygons having weight at most 2/3 the weight of P. In the case of rectilinear polygons, rectilinear cuts having the above properties exist, however, the fraction is 3/4 instead of 2/3. We present examples showing that these bounds are tight in the worst case.
We show that in $O(n)$ time using $O(\log n)$ cuts, a simple polygon can be partitioned into two groups G_1 and G_2 of pieces, such that the ratio of the area of G_1 to the area of G_2 is $x : y$ for any $x, y > 0$, G_1 can be made a single piece, and all but possibly one of the cuts are diagonals of the polygon.
Finally, we present an $O(n)$ time algorithm for finding the shortest chord in a convex polygon P that cuts off α times the area of P.

1 Introduction

Inspired by Chazelle's technique on polygon cutting [Cha82], we explore several variants of his result. The original theorem states the following: Let P be simple polygon on n vertices with each vertex assigned a weight $\in \{0,1\}$, and let W, the weight of P, be the added weight of all the vertices. It is possible to find in $O(n)$ time a diagonal ab of P that partitions it into two polygons, each with a weight not exceeding $2W/3$. Chazelle's cutting theorem deals mainly with finding a diagonal that partitions the vertices in an equitable manner. De Berg [dB91] showed that Chazelle's cutting technique can be modified in the rectilinear case[1]

[1] A rectilinear polygon is a polygon whose edges are axis-parallel. Rectilinear polygons have also been called *orthogonal polygons*, *isothetic polygons* and *rectanguloid polygons* in the literature.

with the fraction being 3/4 rather than 2/3. Our approach has been to explore variants to these polygon cutting theorems by shifting the weight onto different parts of the polygon such as the boundary or the interior.

The present paper is concerned with providing efficient algorithms for cutting two or three dimensional discrete domains with straight line or planar cuts. By discrete domain, we mean simple polygons in the plane, or terrains in three dimensional space. The goal is to obtain corresponding pieces whose area, volume, length of perimeter, etc., attain a desired ratio.

Our approach is to take into account efficiency parameters like the length of cuts or the number of cuts. We provide algorithms that achieve the required cuts and whose running time is expressed in terms of the number of vertices of the discrete domain. Cutting problems find applications in many different areas such as computer-aided manufacturing and design, image processing, pattern recognition or VLSI design (see [KS85] for a survey of polygon cuttings and their applications). Related problems with applications to cloth manufacturing have been considered by Milenkovic et al. [LM93,MDL92]. The problem of bisecting the area of a polygon in relation to the *Ham-sandwich* problem was studied by Díaz and O'Rourke [DO91], Stojmenović [Sto91] and Shermer [She92].

We also study these problems in the rectilinear setting. A *rectilinear polygon* [2] is one whose edges are all aligned with a pair of orthogonal coordinate axes, which we take to be horizontal and vertical without loss of generality. Rectilinear polygons are commonly used as approximations to arbitrary simple polygons; and they arise naturally in domains dominated by Cartesian coordinates, such as raster graphics, VLSI design, robotic, or architecture. A rectilinear polygon is called *trapezoided* if both its vertical and horizontal visibility maps are given. de Berg [dB91] presented the rectilinear version of Chazelle's polygon cutting theorem, which led to several divide-and-conquer algorithms inside a rectilinear polygon, such as rectilinear link query, rectilinear link diameter and rectilinear link center [dB91]. An outline of the paper is as follows. In section 2, we consider algorithms for cutting the boundary of a simple polygon. In section 3, we consider cutting the interior of the polygon. In section 4, we address the rectilinear version of these cutting problems. In section 5, the problem of cutting a simple polygon into two parts of a given ratio is considered. Finally, in section 6, we address the problem of finding the shortest chord in a convex polygon that cuts off a given area.

2 Cutting the boundary of a simple polygon

Let P be simple polygon with vertices $v_0, \ldots, v_{n-1}$ in clockwise order. All index manipulation is modulo n. In this section, we prove the following theorem:

Theorem 1. *Let P be a simple polygon with n vertices $v_0, \ldots, v_{n-1}$ each edge assigned a non-negative weight w_i. Let $W(P)$ denote the sum of the weights on*

[2] Rectilinear polygons have also been called *orthogonal polygons*, *isothetic polygons* and *rectanguloid polygons* in the literature.

the edges. There exists a vertex v_i and a point b on the boundary of P such that the segment v_ib partitions P into two polygons P_1, P_2 satisfying: $W(P_1) \leq W(P_2) \leq 2W(P)/3$. The segment v_ib is called a valid segment.

The weights of the edges in P_1 and P_2 are the same as in P except if an edge of P is partitioned by the segment v_ib. In this case, the weight is distributed according to the ratio of the split.

Before proving Theorem 1, we need the following lemma. An ear of a triangulated polygon P is a triangle where at least two of the edges of the triangle are boundary edges. It is well known (see in [O'R94]) that every triangulated polygon on more than 3 vertices has at least two ears.

Lemma 1. *Given a triangulated polygon P, one of the following holds: (1) There is a diagonal that partitions P into two polygons P_1, P_2 such that $W(P_1) \leq W(P_2) \leq 2W(P)/3$, (2) there is an ear whose weight is greater than $W(P)/3$, (3) there is an edge whose weight is greater than $W(P)/3$.*

Proof. If P is a triangle, then the lemma follows trivially. Consider the case where P has more than three vertices.

A diagonal that partitions a polygon P into two polygons P_1, P_2 such that $W(P_1) \leq W(P_2) \leq 2W(P)/3$ is called a *valid diagonal*. An edge whose weight is greater than $W(P)/3$ is called a *heavy edge*. An ear whose weight is greater than $W(P)/3$ is called a *heavy ear*.

Assign a zero weight to each diagonal in the triangulation of P. Let v_iv_j be an arbitrary diagonal of P. If it is valid, then the lemma follows. Suppose v_iv_j is not valid. This means that either $W(P_1)$ or $W(P_2)$ is greater than $2W(P)/3$. Let P_1 be the polygon $v_i, v_{i+1}, \ldots, v_j, v_i$, and suppose, without loss of generality, that $W(P_1) > 2W(P)/3$. Let v_k be the vertex in P_1 adjacent to v_i and v_j in the triangulation of P. If one of the following conditions hold then the lemma follows.
Condition 1: Either v_iv_k or v_jv_k is valid.
Condition 2: Either v_iv_k or v_jv_k is a heavy edge.
Condition 3: The triangle $v_iv_jv_k$ is a heavy ear.

If none of the three conditions hold, then either v_iv_k or v_jv_k or both are non-valid diagonals. If v_iv_k (resp. v_jv_k) is a diagonal, it partitions P into two polygons. Let P_i (resp. P_j) be the polygon not containing triangle $v_iv_jv_k$.

Assume, without loss of generality, that v_iv_k is a non-valid diagonal. Now, v_jv_k is either an edge of P or a diagonal. If it is an edge, it cannot be heavy which means that the weight $W(P_i)$ is greater than $W(P)/3$. If it is a diagonal, let $W(P_j)$ represent it's weight. We now show that weight $W(P_i)$ or $W(P_j)$ is greater than $2W(P)/3$.

Since $W(P_1)$ is greater than $2W(P)/3$ and the weight of v_iv_j is zero, the sum $W(P_i) + W(P_j) > 2W(P)/3$. This implies that either $W(P_i)$ or $W(P_j)$ has weight greater than $W(P)/3$. Suppose, without loss of generality, $W(P_i)$ has weight greater than $W(P)/3$. Since v_iv_k is not valid, this means that $W(P_i)$ must also be greater than $2W(P)/3$. Note that P_i has fewer vertices than P_1. Since P has a finite number of vertices, by repeating this argument we eventually must terminate with a triangle. But a triangle is an ear of P, therefore, the lemma follows.

We are now in a position to prove Theorem 1.

Proof of Theorem 1: By Lemma 1, if P has a valid diagonal, then it also has a valid segment. If P has a heavy ear, then the ear can be partitioned by a segment such that the segment is valid. Finally, if P has a heavy edge, the edge can be partitioned by a segment such that the segment is valid. Since one of these three conditions must exist, by Lemma 1, the theorem follows. $\square$

Given a triangulated polygon, a valid segment can be computed in linear time.

Corollary 1. *Let P be a simple polygon with n vertices $v_0, \ldots, v_{n-1}$ and each edge assigned a non-negative weight w_i. A valid segment of P can be computed in $O(n)$ time.*

By assigning a weight equal to the length of an edge, we have the following corollary.

Corollary 2. *Let P be a simple polygon with n vertices $v_0, \ldots, v_{n-1}$. There exists a vertex v_i and a point b on the boundary of P such that the segment $v_i b$ partitions the perimeter of P into two pieces P_1, P_2 satisfying: $|P_1| \leq |P_2| \leq 2|P|/3$.*

Note that when the weights are assigned to vertices, there always exists a diagonal that can partition the polygon, however, when the weights are assigned to edges, one of the two end points of a splitting segment no longer needs to be a vertex. See Figure 1 for an example. Figure 3 is an example showing that the bound of 2/3 is tight (by placing weights 1/3,0,0,1/3,0,0,1/3,0,0 on the edges in clockwise order).

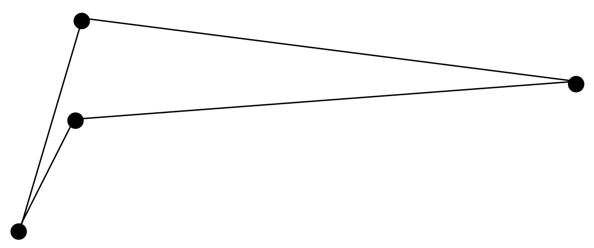

Fig. 1. A polygon that does not have a valid diagonal cut in terms of boundary or area.

3 Cutting the interior of a simple polygon

In this section, we shift the weight from the edges to the interior. We study the special case where weight is defined by area, then we show that the weight function can be generalized. We first prove the following theorem:

Theorem 2. *Let P be simple polygon with vertices $v_0, \ldots, v_{n-1}$. Let $A(P)$ denote the area of P. There exists a vertex v_i and a point b on the boundary of P such that the segment $v_i b$ partitions P into two polygons P_1, P_2 satisfying: $A(P_1) \leq A(P_2) \leq 2A(P)/3$. The segment $v_i b$ is called a* valid segment.

Before proving Theorem 2, we need the following lemma.

Lemma 2. *Given a triangulated polygon P, one of the following holds: (1) There is a diagonal that partitions the polygon into two polygons P_1, P_2 such that $A(P_1) \leq A(P_2) \leq 2A(P)/3$, (2) there is a triangle in P's triangulation whose area is greater than $2A(P)/3$, (3) there is a triangle in P's triangulation that partitions P into three pieces with each piece having area less than $A(P)/3$.*

Proof. If P is a triangle, then the lemma follows. Consider the case where P has more than three vertices.

A diagonal that partitions a polygon P into two polygons P_1, P_2 such that $A(P_1) \leq A(P_2) \leq 2A(P)/3$ is called a *valid diagonal*. A triangle whose weight is greater than $2A(P)/3$ is called a *heavy triangle*.

Let $v_i v_j$ be an arbitrary diagonal of P. If it is valid, then the lemma follows. Suppose $v_i v_j$ is not valid. This means that either $A(P_1)$ or $A(P_2)$ is greater than $2A(P)/3$. Let P_1 be the polygon $v_i, v_{i+1}, \ldots, v_j, v_i$, and suppose, without loss of generality, that $A(P_1) > 2A(P)/3$. Let v_k be the vertex in P_1 adjacent v_i and v_j in the triangulation of P. If one of the following conditions hold then the lemma follows.

Condition 1: Either $v_i v_k$ or $v_j v_k$ is valid.
Condition 2: Triangle $v_i v_j v_k$ is a heavy triangle.
Condition 3: Triangle $v_i v_j v_k$ is a triangle that partitions P into three pieces such that the weight of each piece is less than $A(P)/3$.

If none of the three conditions hold, then either $v_i v_k$ or $v_j v_k$ or both are non-valid diagonals. If $v_i v_k$ (resp. $v_j v_k$) is a diagonal, it partitions P into two polygons. Let P_i (resp. P_j) be the polygon not containing triangle $v_i v_j v_k$.

Either $A(P_i)$ or $A(P_j)$ is greater than $2A(P)/3$. Suppose, without loss of generality, that $A(P_i) > 2A(P)/3$. Note that P_i has fewer vertices than P_1. Since P has a finite number of vertices, this process must terminate with a triangle. But such a triangle would be heavy, therefore, the lemma follows.

We are now in a position to prove Theorem 2.

Proof of Theorem 2: If P has a valid diagonal, then the theorem follows.

Suppose P has a heavy triangle $v_i v_j v_k$. Let a be a point on $v_i v_j$ with the property that $v_k a$ divides the weight of triangle $v_i v_j v_k$ equally. Let b be the first intersection point of a ray emanating from v_k in direction of a with the boundary of P. Segment $v_k b$ is a valid segment.

Finally, suppose P has a triangle $v_i v_j v_k$ that partitions P into three pieces P_1, P_2, P_3 such that the weight of each piece is less than $A(P)/3$. Let P_1, P_2, P_3 be adjacent to $v_i v_k$, $v_k v_j$, $v_j v_i$ respectively. Let a be a point on $v_i v_j$ with the property that $v_k a$ divides the weight of triangle $v_i v_j v_k$ such that $A(P_1)$ and the area of triangle $v_i v_k a$ is equal to $A(P)/3$. Such a point must exist since the weight of $A(P_1)$ and triangle $v_i v_j v_k$ is greater than $A(P)/3$. Let b be the first intersection point of a ray emanating from v_k in direction of a with the boundary of P. Segment $v_k b$ is a valid segment since the weight of P_3 is less than $A(P)/3$.

Since one of these three conditions must hold, by Lemma 2, the theorem follows. □

Observe that the inductive proof of Lemma 2 implies walking along a path of the dual graph of the triangulation tree of P. Since each step of this walk takes O(1) time, we have the following.

Corollary 3. *Let P be a simple polygon with n vertices $v_0, \ldots, v_{n-1}$. A valid segment of P can be computed in $O(n)$ time.*

Theorem 2 extends naturally in the following way. Let $z = f(x, y)$ be a continuous, non-negative function defined over the domain of a simple polygon P lying in the plane. Let V be the volume of the cylinder, bounded from above by the surface defined by function f and bounded from below by polygon P.

Corollary 4. *There exists a vertical plane, cutting P along a chord, which partitions V into two parts, each one of volume greater than $V/3$ and smaller than $2V/3$.*

Instead of a function f we can also take an infinitely additive measure defined for the space of triangles in the plane.

3.1 The Discrete Case

We now consider the case where P has k points $\{p_0, p_1, \ldots, p_{k-1}\}$ in its interior. Each p_i is assigned a non-negative weight w_i. Let $W(P)$ represent the sum of the weights of the points $p_i \in P$.

Theorem 3. *There exists a vertex v_i and a point b on the boundary of P such that the segment $v_i b$ partitions P into two polygons P_1, P_2 satisfying: $W(P_1) \leq W(P_2) \leq 2W(P)/3$. The segment $v_i b$ is called a* valid segment.

The weights of the points in P_1 and P_2 are the same as in P, except for the points on $v_i b$, whose weights are not considered (i.e. considered to be zero) since they do not lie in the interior of either polygon. Lemma 2 still holds in this discrete setting. To compute a valid segment $v_i b$, one can first construct the triangulation of P and for each point p_i determine the triangle containing it.

This may be done using $O(\log n)$ point location for each point $p_i (i = 0, 1, \ldots, k-1)$ after $O(n)$ time preprocessing. Hence the total complexity of this step is

$O(n + k \log n)$. In order to meet one of the conditions of Lemma 2, walk along a path of the dual graph of the triangulation of P. This path has length at most $O(n)$. Contrary to the case of cutting the area, in the discrete case we spend $O(k)$ time accumulating weights of the points falling inside the triangles on the path. This process takes $O(n + k)$ time. Finally, as in Theorem 2, we have to make a cut through a triangle determined in condition 1 or condition 2 of Lemma 2 which can be done in $O(k)$ time using prune and search.

This triangle may contain $O(k)$ points. In this triangle, we have to make a cut through a vertex v of P, dividing the points such that the weights on either side sum up to a specific value. This may be easily done in $O(k \log k)$ time by performing a rotational sweep around v after sorting the points. However, using prune and search, we can perform the last step in $O(k)$ time. Hence the total complexity of the algorithm remains $O(n + k \log n)$. We conclude with the following theorem.

Theorem 4. *The valid segment of polygon P with n vertices and k weighted points inside P may be found in $O(n + k \log n)$ time.*

4 Cutting a rectilinear polygon

A rectilinear polygon is a simple polygon having all edges axis-parallel. In the case of rectilinear polygons, we are interested in finding valid axis-parallel cuts. The bounds in Chazelle's polygon cutting theorem as well as Theorems 1, 2, and 3 all hold provided that no pair of edges lies on the same vertical line. However, the general position assumption (no three vertices collinear) usually does not hold in the rectilinear setting. When this assumption is removed, de Berg [dB91] showed that a valid rectilinear cut still holds with slightly modified bounds when a non-negative weight is assigned to each vertex. De Berg proves the following: Let P be a simple rectilinear polygon with n vertices each assigned a non-negative weight w_i and $W(P)$ is the added weight of all the vertices. Then there exists a cut segment (i.e. an axis-parallel segment connecting two edges of P and lying entirely in P) that partitions P into two rectilinear polygons each of weight as most $3/4\ W(P)$. Moreover, the segment can be chosen such that it is incident upon at least one vertex and can be computed in $O(n)$ time. See Figure 2 for de Berg's example showing that this bound is tight in the worst case.

The arguments presented in Section 2 and Section 3 can be modified to show that similar bounds hold when the weight is shifted to the boundary or the interior. In the previous sections, we showed that by traversing a triangulation of a simple polygon, a valid segment can be computed. In the rectilinear setting, by traversing the horizontal and vertical trapezoidalization of the polygon, a valid axis parallel cut can be computed (see [O'R94] for a reference on trapezoidalization.

We conclude this section by stating the resulting theorems. Let P be a rectilinear polygon with n vertices. Each edge is assigned a non-negative weight w_i. Let $W(P)$ denote the sum of the weights on the edges.

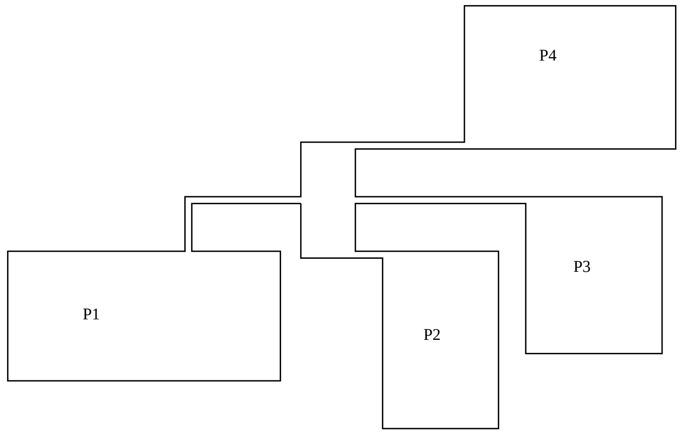

Fig. 2. A rectilinear polygon in which every horizontal or vertical cut results in a polygon of weight at least $3/4\ W(P)$.

Theorem 5. *There exists two points a and b on the boundary of P such that segment ab is axis-parallel and partitions P into two polygons P_1, P_2 satisfying: $W(P_1) \le W(P_2) \le 3W(P)/4$.*

Theorem 6. *Let $A(P)$ denote the area of P. There exists two points a and b on the boundary of P such that the segment ab is axis-parallel and partitions P into two polygons P_1, P_2 satisfying: $A(P_1) \le A(P_2) \le 3A(P)/4$.*

We now consider the case where P has k points $\{p_0, p_1, \ldots, p_{k-1}\}$ in its interior. Each p_i is assigned a non-negative weight w_i. Let $W(P)$ represent the sum of the weights of the points $p_i \in P$.

Theorem 7. *There exists two points a and b on the boundary of P such that the segment ab is axis-parallel and partitions P into two polygons P_1, P_2 satisfying: $W(P_1) \le W(P_2) \le 3W(P)/4$.*

Computing a horizontal and vertical trapezoidalization can be acheived in linear time. Therefore, finding these valid axis-parallel cuts can be done in linear time.

5 Cutting a simple polygon into equal parts

Suppose that we want to partition a polygon into two equal area pieces. Clearly, it is not always possible to find a diagonal to achieve this (see Figure 1 for an example). In fact, this may not be possible to achieve with a single chord (see Figure 3 for such an example).

In this section, we show that using $O(\log n)$ cuts where n is the number of vertices of the polygon, we can partition a polygon into two groups G_1 and G_2 of

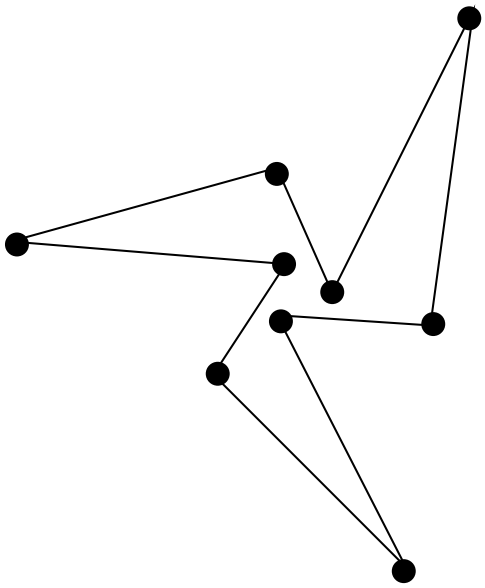

Fig. 3. A polygon that does not have a single chord partitioning it into two equal area parts.

pieces such that any ratio can be achieved between the areas of the two groups. In fact, in our cutting scheme either one of G_1 or G_2 can be made to remain a single piece and all but possibly one of the cuts are diagonals of the polygon.

Let P be a simple polygon on n vertices.

Theorem 8. *With at most $\lfloor \log_{(3/2)} n \rfloor$ cuts, P can be partitioned into two groups G_1 and G_2 of pieces, such that the ratio of the area of G_1 to the area of G_2 is $x : y$ for any $x, y > 0$, G_i can be made a single piece, and all but possibly one of the cuts are diagonals of the polygon.*

Proof: We proceed by induction on the number of vertices of P. If P is a triangle, the theorem holds trivially.
Inductive Hypothesis: Given a polygon P on $n \leq k, k \geq 3$ vertices, with at most $\lfloor \log_{(3/2)} n \rfloor$ cuts, P can be partitioned into two groups G_1 and G_2 of pieces, such that the ratio of the area of G_1 to the area of G_2 is $x : y$ for any $x, y > 0$, G_i can be made a single piece, and all but possibly one of the cuts are diagonals of the polygon.
Inductive Step: Let P be a polygon on $n = k + 1$ vertices and let $x : y$ be the ratio to be achieved. Let $A(P)$ be the area of P. We want to partition P into two groups such that one piece has area $xA(P)/(x + y)$ and the other has area $yA(P)/(x + y)$.

Using Chazelle's algorithm, we can find a diagonal of P that partitions P into two polygons P_1 and P_2 where each P_i has at most $2/3$ of the vertices of P.

Suppose that P_1 has area $xA(P)/(x+y)+\epsilon$ for some $\epsilon > 0$. We can apply the inductive hypothesis to cut P_1 into ratio $xA(P)/(x+y) : \epsilon$, with the piece of size $xA(P)/(x+y)$ remaining intact. Since P_1 has at most 2/3 the vertices of P, the theorem follows.

If on the other hand P_1 has area $xA(P)/(x+y) - \epsilon$, then we apply the inductive hypothesis on P_2. We cut P_2 into two pieces with the single piece having area ϵ being adjacent to P_1. Thus, we end up with the desired partition and the theorem follows. □

The inductive proof gives rise to a linear time algorithm for finding these cuts. Initially, the given polygon P consists of n vertices. At each invocation of Chazelle's algorithm[Cha82], the remaining piece to be cut has at most 2/3 of the original vertices. This gives rise to the following recurrence for the running time: $T(n) \leq T(2n/3) + O(n)$, which solves to $O(n)$.

6 Finding the shortest α-chord of a convex polygon

As we have seen in the previous section, there does not necessarily exist a chord that bisects the area of a simple polygon into two equal area parts, however, such a chord always exists in a convex polygon. By performing a simple binary search from any vertex of the convex polygon, one can find such a chord in $O(\log n)$ time. In this section, we address the problem of finding the shortest such chord. In $O(n)$ time we show how to find the shortest chord that cuts a convex polygon into two pieces of any given ratio.

Let $P = \{v_0, v_1, \ldots, v_{n-1}\}$ be a convex polygon on n vertices and let α represent the fraction of P to remove. A chord that removes α times the area P shall be referred to as an α-chord. We first show how to solve the special case where $\alpha = 1/2$. Then we show how to generalize this approach to handle any $0 < \alpha < 1$. A chord that divides the area of a polygon into half is called an *area bisector*.

Let n_i represent the outer-normal of the edge e_i of P. Suppose the shortest area bisector has endpoints on the interior of edges e_j and e_k of P. Let N represent one of the two normals of the line containing the shortest area bisector. Then, the following can be proved.

Lemma 3. *The dot product $n_j \cdot N$ is equal to $n_k \cdot N$.*

Given a pair of edges of P, there can be at most one chord having endpoints on the interior of these edges that bisects the polygon and satisfies the criterion of Lemma 3. Therefore, shortest area bisector can be found by examining every pair of edges. However, this would imply a quadratic algorithm. In order to reduce the number of possible candidates, we first observe the following:

Property 1. A vertex of a convex polygon is part of a unique area bisector.

Let x_i represent the endpoint of the unique area bisector emanating from v_i.

Lemma 4. *If the shortest area bisector has one endpoint on the part of the boundary from x_i to x_{i+1}, the other endpoint must be on the edge from v_i to v_{i+1}*

Lemma 4 implies that there are $O(n)$ candidates that need to be verified in the search for the shortest area bisector and Lemma 3 implies that each candidate can be tested in $O(1)$ time.

Theorem 9. *The shortest area bisector of a convex polygon with n vertices can be found in $O(n)$ time.*

We now turn our attention to finding the shortest chord that cuts a fraction α of the polygon for any $0 < \alpha < 1$. The main observation is that a vertex no longer is part of a unique α-chord when α is not $1/2$, a vertex is part of exactly two α-chords. Let x_i and y_i represent the endpoints of the two α-chords emanating from v_i. If we consider the chords to be directed from v_i, then one of the chords cuts off an α fraction of P to its left and the other cuts off an α fraction to its right. A directed chord which cuts off a fraction of α to its left (resp. right) will be called a left (resp. right) α-chord. WLOG, let $v_i x_i$ represent all the left α-chords and $v_i y_i$ represent all the right α-chords.

Lemma 5. *If the shortest left (resp. right) α-chord has one endpoint on the part of the boundary from x_i to x_{i+1} (resp. y_i to y_{i+1}), the other endpoint must be on the edge from v_i to v_{i+1}*

Lemma 5 implies that there are $O(n)$ candidates that need to be verified in the search for the shortest area bisector and Lemma 3 implies that each candidate can be computed in $O(1)$ time.

Theorem 10. *The shortest α-chord of a convex polygon with n vertices can be found in $O(n)$ time.*

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