Algorithms for Packing Two Circles in a Convex Polygon

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Abstract. We consider algorithms for packing two circles in a given n-vertex convex polygon. The first algorithm outputs a pair of equal non-overlapping circles of maximum radius and has running time $O(n)$. The second algorithm outputs a pair of non-overlapping circles whose sum of radii is maximum and has running time $O(n^2)$.

1 Introduction

We consider the following packing problems:

Problem 1. Compute the maximum radius and placement of two equal circles that can be packed inside an n-vertex convex polygon.

Problem 2. Compute the maximum sum of radii and placement of two circles that can be packed inside an n-vertex convex polygon.

In this paper we give algorithms with running times $O(n)$ and $O(n^2)$ for solving Problems 1 and 2, respectively. We note that for the case of a single circle the problem can be solved in linear time using the medial axis diagram of the given convex polygon ([6, 7, 1]).

We note that Biedl et al. [2], independently, presented an $O(n^2)$ time algorithm to solve Problem 1 in the context of folding polygons. In fact, they address the more general problem of placing two equal circles in a simple polygon. It is also worth pointing out that Problem 1 has been investigated in the context of finding the maximum radius of k equal circles that can be packed in a unit square [4]. There is also a related "dual" to our packing problem which is known in the literature as the k-center problem [8, 9]: given a set of n points (resp. a simple polygon) in the plane find k congruent circles of smallest radius whose union covers the set (resp. the simple polygon).

1.1 Preliminaries

An important concept in this paper is the *medial axis diagram* of a convex polygon P: this is the locus of centers of maximal circles in P (i.e., circles inside

P that cannot be enclosed in any other circle inside P). The medial axis of an n-vertex convex polygon can be constructed in linear time [1]. Vertices of the medial axis are centers of circles tangent to at least three distinct edges of the polygon, while segments of the medial axis form the locus of the center of a circle tangent to at least two given edges of the polygon. We also note that the medial axis diagram of a convex polygon is a tree of size $O(n)$. This is true even for non-convex (simple) polygons, although in this case some segments of the medial axis diagram may be curved (parabolic) lines.

The medial axis diagram of a convex polygon gives rise to *virtual vertices*: these are formed by the intersection of pairs of edges (not necessarily adjacent) of the polygon which correspond to a segment of the medial axis diagram. We note that all the vertices of the convex polygon are also virtual vertices. An

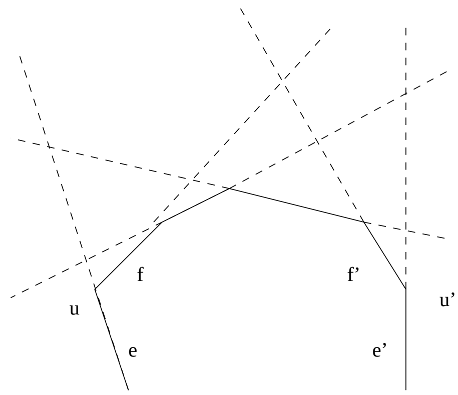

Fig. 1. A convex polygon and possible virtual vertices determined by the intersection of edges.

example of a convex polygon and some (not all) virtual vertices is depicted in Figure 1.

In order to simplify some of the arguments to follow, we will assume that the polygon has no pair of parallel edges.

Another important notion is the *antipodal pair* of vertices virtual or otherwise of a convex polygon P: this is a pair of such vertices which admit parallel lines of support for P. We recall (see [7]) that a convex n-gon has a linear number of such pairs.

1.2 Outline of the paper

Here is an outline of the paper. Section 2 solves Problem 1, and Section 3 considers Problem 2. In Section 4 we discuss the performance of the greedy algorithm. We summarize and conclude in Section 5.

2 Packing Two Equal Circles

We consider Problem 1 for convex polygons. A circle is called *feasible* if it is enclosed in the given polygon and its perimeter is tangent to at least two edges of the polygon.

To start, we prove a useful lemma for the case of the triangle.

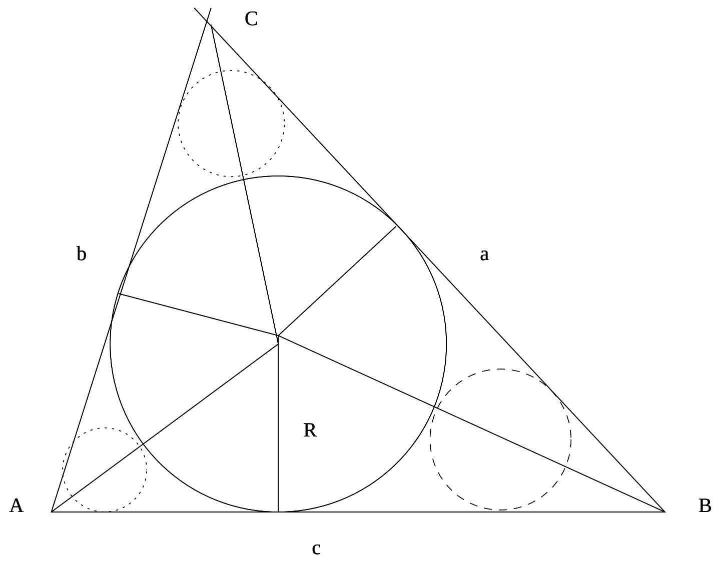

Fig. 2. Maximum radius (solid) circle enclosed in a triangle. The solid and dashed circles are also the output of the greedy algorithm.

Lemma 1. *The maximum radius circle enclosed in a triangle is centered at the intersection of the three bisectors of the triangle and can be computed in constant time.*

PROOF We refer to Figure 2. Let the triangle be ABC with vertices A, B, C and corresponding sides a, b, c opposite to, and angles α, β, γ adjacent to vertices A, B, C, respectively. The desired circle is centered at the intersection of the three bisectors of the triangle and (using the well-known law of sines) its radius R is easily computed by the formula

$$R = \frac{\sin(\alpha/2)\sin(\beta/2)}{\cos(\gamma/2)}c.$$

Details of the proof are left to the reader. □

The next lemma is important in our subsequent discussions and is easy to prove from the definitions.

Lemma 2. *To maximize the radius of two identical disjoint circles enclosed in a convex polygon, each circle must be feasible. Moreover, unless both circles are tangent to a pair of parallel edges of P they must also be tangent to each other.*

The next lemma identifies an invariance property of our problem.

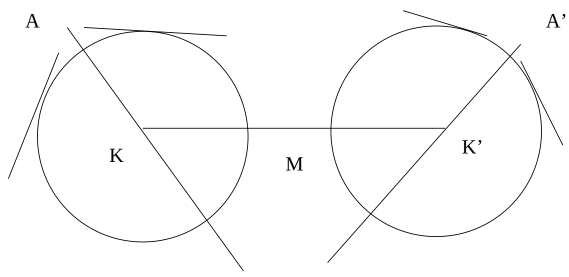

Fig. 3. A feasible pair of circles with centers K, K'.

Lemma 3. *Consider a pair of feasible non-overlapping circles in the convex polygon P, with centers K, K', respectively, and tangent to its perimeter along edges of the polygon (see Figure 3). As the centers K, K' traverse two given medial axis lines the midpoint M of the line segment connecting the centers K and K' must lie on a straight line.*

PROOF The centers K, K' of the two circles are moving along medial axis lines emanating from the corresponding edges. The result will follow from a general claim. For the proof we refer to Figure 4. Without loss of generality we may assume that the corresponding positions of the centers of the circles are

$$\begin{array}{l} A = (0,0)\ \ B = (0,a)\ \ C = (0,ra) \\ A' = (b,c)\ B' = (d,e)\ C' = (b + r(d-b), c + r(e-c)). \end{array}$$

The midpoints of the line segments AA', BB', CC' are

$$\begin{array}{l} M = (b/2, c/2), \\ N = (d/2, (a+e)/2), \\ O = ((b + r(d-b))/2, (c + r(e-c) + ra)/2), \end{array}$$

respectively. In view of [7] in order to prove that the points M, N, O are collinear it is enough to show that

$$\det \begin{bmatrix} (b + r(d-b))/2 & (c + r(e-c) + ra)/2 & 1 \\ d/2 & (a+e)/2 & 1 \\ b/2 & c/2 & 1 \end{bmatrix} = 0$$

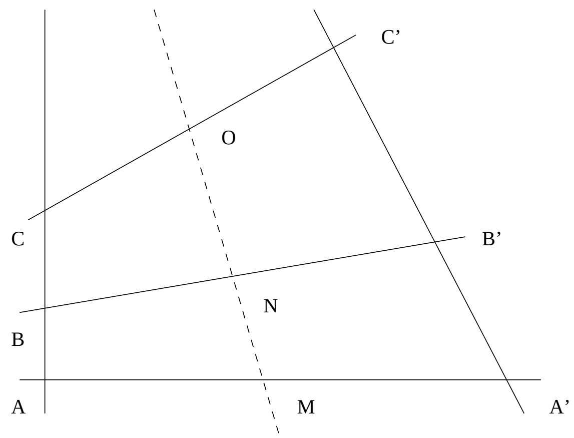

Fig. 4. The midpoints of the line segments joining the centers of feasible pairs of circles must lie on a straight line.

To see this we argue as follows. We factor $1/2$ from the first and second column of the determinant. Next we substract the third row from the second one and multiply the result by r. The resulting determinant is

$$\det \begin{bmatrix} b + r(d-b) & c + r(e-c) + ra & 1 \\ r(d-b) & r(e+a-c) & 0 \\ b & c & 1 \end{bmatrix}$$

If we substract the second row from the first, then the first and third row of the resulting determinant are identical. Hence the determinant must be equal to 0, as desired. □

Lemma 4. *If a pair of identical, feasible, non-overlapping circles is optimal then both circles must lie on medial axis lines whose corresponding vertices virtual or otherwise form an antipodal pair.*

Proof Consider an optimal pair of identical feasible circles. In view of our previous discussion we may assume without loss of generality the circles are feasible and tangent to each other. Assume on the contrary the two circles do not lie on a medial axis segment generated by an antipodal pair of virtual vertices. It follows that the region determined by edges of virtual vertices corresponding to the two circles must be unbounded. It follows that we can enlarge both circles by sliding them towards the unbounded region (see Figure 5). This contradicts the maximality of the pair of circles. □

Now we are in a position to prove the main theorem.

Theorem 1. *Given an n-vertex convex polygon P we can determine in time $O(n)$ the maximum radius of two equal non-overlapping circles enclosed inside P.*

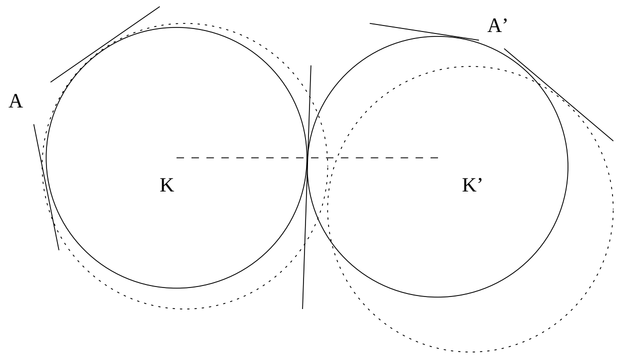

Fig. 5. If the vertices A, A' are not antipodal a better solution can be found.

Proof By Lemma 4, an optimal solution must lie on a pair of medial-axis lines corresponding to an antipodal pair of virtual vertices. By [7] there is a linear number of pairs of antipodal vertices and in linear time we can find a pair. Start with any such pair of antipodal vertices. View them as leaves of the medial axis tree and walk along the tree from child to parent. In time linear in the number of vertices of the path connecting the two vertices we can determine a pair of virtual vertices on which a pair of feasible circles is possible. This is achieved by traversing vertices of the medial-axis tree from child to parent simultaneously starting from both leaves until two medial-axis vertices are found for which the resulting circles intersect (possibly overlapping).

Let (A, A') be such a pair of vertices (virtual or otherwise). By Lemma 3 the set of midpoints of the line segments joining the centers of pairs of feasible non-overlapping circles tangent to the edges of A and A' lie on a straight line. This line is determined by just two points. One such point is the midpoint M of the line segment connecting A and A'. A second such point N is the midpoint of the line segment joining the centers of a feasible pair of circles as above. Each of the desired circles is delimited by the line MN and the edges adjacent to A, respectively A', being the largest circle enclosed in the corresponding triangle (see Lemma 1). If the two resulting circles are tangent to each other the pair is stored.

We select another leaf (i.e., vertex of the polygon) which is antipodal to any of the two leaves previously considered and repeat the process of the previous paragraph. We observe that each transition increases the size of the resulting circles, hence paths of the medial-axis tree previously traversed need not be traversed again.

Since the medial-axis tree is of linear size, the running time for the construction of all these pairs of circles is linear. Ultimately the optimal pair is determined among the stored solutions. This completes the proof of the theorem. $\square$

3 Maximizing the Sum of Radii

In this section we consider Problem 2 for convex polygons, which concerns the more general case of two circles (not necessarily equal) with maximum sum of radii. The first lemma is a straightforward extension of Lemma 4 to the case of two arbitrary circles.

Lemma 5. *If a pair of feasible, non-overlapping circles is optimal then both circles must lie on medial axis lines whose corresponding vertices virtual or otherwise form an antipodal pair.* □

Lemma 6. *Assume that the centers K and K' of two circles move along two medial axis lines, respectively, of the medial axis diagram, always remaining in contact, as well as feasible. If the ratio of the rate of change of the radius of K over the rate of change of the radius of K' is constant, then for any fixed time interval the ratio of the displacement of the center K over the displacement of the center K' is constant.*

Proof For the proof we refer to Figure 6. Let $L(r), L(r')$ and $K(R), K(R')$ be

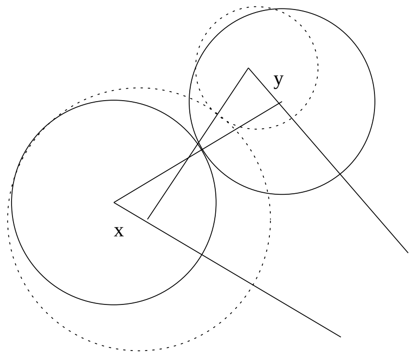

Fig. 6. Rate of change of the displacement of the centers.

the distance from A of the centers with radius r, r' and the distance from B of the centers with radius R, R'. By assumption the ratio $(R - R')/(r - r')$ is constant. Moreover, $R = L(R)\sin(\alpha/2), R' = L(R')\sin(\alpha/2)$ and $r = K(r)\sin(\beta/2), r' = K(r')\sin(\beta/2)$, where α, β are the angles at A, B. If $x = L(R) - L(R')$ and $y = K(r) - K(r')$ are the two displacements, respectively, then it follows that

$$\begin{aligned}\frac{x}{y} &= \frac{L(R)-L(R')}{K(r)-K(r')}\\ &= \frac{R/\sin(\alpha/2)-R'/\sin(\alpha/2)}{r/\sin(\beta/2)-r'/\sin(\beta/2)}\\ &= \frac{R-R'}{r-r'}\frac{\sin(\beta/2)}{\sin(\alpha/2)}\end{aligned}$$

This proves the lemma. $\square$

Lemma 7. *Let L, L' be two medial axis lines. The centers of two circles traverse the lines L, L' as in Lemma 6. Let A, A', A'' and B, B', B'' be three points located on lines L and L', respectively, as depicted in Figure 7. Then the line segment $A'B'$ cannot be the longest of the three segments $AB, A'B', A''B''$.*

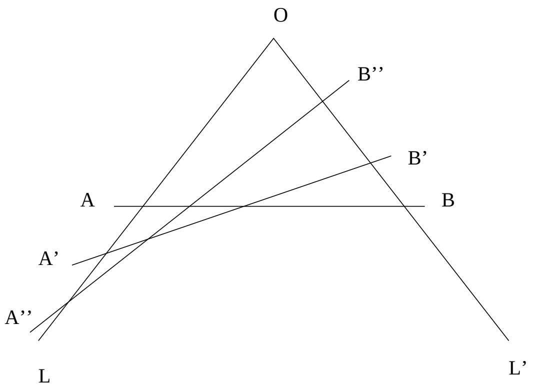

Fig. 7. The line segment $A'B'$ cannot be the longest of the three segments $AB, A'B', A''B''$.

PROOF Without loss of generality we may assume that the lines L, L' are intersecting at a point O. (Note that if the lines are parallel the conclusion of the lemma is easy.) Let ϕ be the angle AOB between lines L, L' at O. Let $\alpha, \beta, \alpha', \beta'$ be the angles $OAB, OBA, OA'B', OB'A'$, respectively. Let $a = |OA|, b = |OB|, x = |AA'|, y = |BB'|$. Furthermore, let s, s' be the lengths of the line segments $AB, A'B'$, respectively. Using the law of sines in the triangle $OA'B'$ we obtain that

$$\frac{s'}{\sin\phi} = \frac{a+x}{\sin\beta'} = \frac{b-y}{\sin\alpha'}.$$

It follows that

$$s' = \frac{a+x}{\sin\beta'}\sin\phi = \frac{b-y}{\sin\alpha'}\sin\phi$$

and

$$(a+x)\sin\alpha' = (b-y)\sin\beta'.$$

By Lemma 6 we may assume without loss of generality that there is a constant $\lambda > 0$ such that $y = \lambda x$. Solving for x we obtain

$$x = \frac{b\sin\beta' - a\sin\alpha'}{\sin\alpha' + \lambda\sin\beta'}.$$

It follows that

$$s' = \frac{\sin\phi}{\sin\beta'}(a+x) = \frac{\sin\phi}{\sin\beta'}\left(a + \frac{b\sin\beta' - a\sin\alpha'}{\sin\alpha' + \lambda\sin\beta'}\right) = \frac{(a\lambda + b)\sin\phi}{\sin\alpha' + \lambda\sin\beta'}.$$

Observe now that $\beta' = \pi - \phi - \alpha'$ and hence $\sin\beta' = \sin(\phi + \alpha')$. It follows that

$$s' = \sin\phi \frac{a\lambda + b}{\sin\alpha' + \lambda\sin(\alpha' + \phi)'}. \tag{1}$$

We now look at s' as a function of α'. It is easy to see that the function

$$F(\alpha') = \sin\alpha' + \lambda\sin(\alpha' + \phi)$$

in the denominator of the right-hand side of (1) (with variable α') has a second derivative which is < 0. This implies that F is concave downwards and hence s' is concave upwards. We see easily that for any three consecutive points on the curve of s' the middle point cannot be a maximum. This concludes the proof of the lemma. □

Lemma 8. *There always exists an optimal solution (i.e a pair of circles whose sum of their radii is maximal) having the center of at least one of the two circles on the vertex of the medial axis diagram.*

Proof We refer to Figure 8. Assume on the contrary that none of the two circles

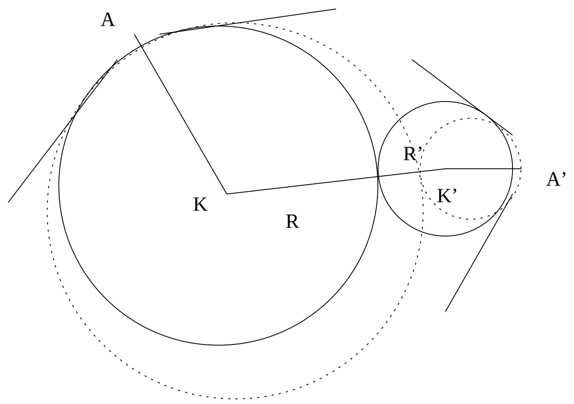

Fig. 8. Improving on the solution by sliding the centers along the medial axis lines.

centered at K, K' lie on vertices of the medial axis diagram. Using Lemma 7 we can improve on the solution depicted in Figure 8. This is done as follows. Since none of the two centers lies on a vertex of the medial axis diagram we can enlarge one of the two circles and at the same time diminish the other by an infinitesimal amount (so as to maintain the requirements of packing). This results into two extra positions for the pair of centers. Lemma 7 implies that one of these two new positions must give a better sum of radii. This is a contradiction because the solution is assumed to be optimal. □

Theorem 2. *Given an n-vertex convex polygon P we can determine in time $O(n^2)$ a pair of non-overlapping circles enclosed inside P whose sum of radii is maximum.*

PROOF In view of Lemma 8 one of the two circles of an optimal pair must be centered on a vertex of the medial axis diagram of the convex polygon. Given such a circle we can pack a second circle of maximum radius in $O(n)$ time by traversing the medial axis tree, which is of size $O(n)$ [1]. The desired optimal pair can then be found in $O(n^2)$ time. This completes the proof of the theorem. □

4 The Greedy Algorithm

It is tempting to consider the performance of the greedy algorithm on Problem 2. Here by greedy algorithm we mean the following:

1. **Pack** a maximal circle inside the convex polygon;
2. **Pack** a second circle which is maximal under the condition that it does not overlap with the first circle;
3. **Output** a pair of circles among these with maximum sum of radii.

This algorithm runs in linear time since both step 1 and step 2 can be performed in linear time. We note that the greedy algorithm does not always lead to an optimal solution (Figure 9 provides an example), however, it will always produce a solution that is at most 1/2 the optimal solution.

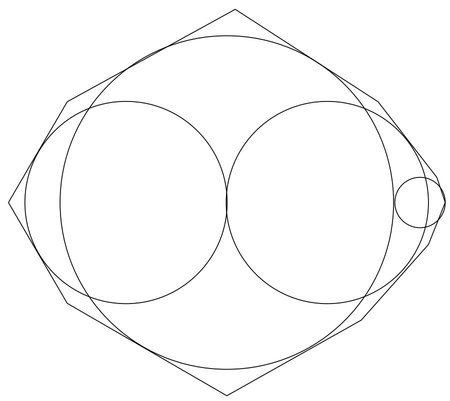

Fig. 9. An example of a convex polygon for which the greedy algorithm is not optimal.

Example 1. For the convex polygon depicted in Figure 9, the sum of the radii of the largest and smallest circles (obtained from the greedy algorithm) is smaller than the sum of the radii of the two equal circles.

At the same time, triangles are examples of convex polygons for which the greedy algorithm is optimal, see Figure 2. For any triangle, the greedy algorithm outputs a pair of circles whose sum of radii is maximum. The claim in this example is easy to prove. By Lemma 5 one of the two circles in an optimal pair of circles must be centered at the intersection of the three bisectors of the triangle.

5 Conclusion and Open Problems

In this paper we considered the problem of packing two circles in convex polygons. We considered separately the case where the circles are equal and where the sum of their radii is maximized. An interesting problem would be to improve the $O(n^2)$ running time of the algorithm of Theorem 2.

We note that the more general problems of packing either (1) k equal circles of maximum radius, or (2) k circles whose sum of radii is maximum, are intractable by the techniques of the paper. One reason appears to be that for $k \geq 3$ none of the centers of the circles of an optimal solution may lie on the medial axis diagram. Another reason is that it would be necessary to develop the notion of medial axis diagram for polygons with "curved" lines. It would also be interesting to look at shapes other than circles, e.g. squares, rectangles, etc.

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