DIAMONDS ARE NOT A MINIMUM WEIGHT TRIANGULATION'S BEST FRIEND

Prosenjit Bose* Luc Devroye† William Evans‡

Abstract

Two recent methods have increased hopes of finding a polynomial time solution to the problem of computing the minimum weight triangulation of a set S of n points in the plane. Both involve computing what was believed to be a connected or nearly connected subgraph of the minimum weight triangulation, and then completing the triangulation optimally. The first method uses the light graph of S as its initial subgraph. The second method uses the LMT-skeleton of S. Both methods rely, for their polynomial time bound, on the initial subgraphs having only a constant number of components. Experiments performed by the authors of these methods seemed to confirm that randomly chosen point sets displayed this desired property. We show that there exist point sets where the number of components is linear in n. In fact, the expected number of components in either graph on a randomly chosen point set is linear in n, and the probability of the number of components exceeding some constant times n tends to one.

*Research supported by NSERC and FIR. Département de mathématiques et d'informatique, Université du Québec à Trois-Rivières, C.P. 500, Trois-Rivieères, Québec G9A 5H7, Canada. This work was carried out while the author was at the Department of Computer Science, University of British Columbia. E-mail: jit@uqtr.uquebec.ca.

†Research supported by NSERC. School of Computer Science, McGill University, Montreal, Canada, H3A 2A7. E-mail: luc@cs.mcgill.ca.

‡Research supported by NSERC. Department of Computer Science, University of British Columbia, 201-2366 Main Mall, Vancouver, B.C., V6T 1Z4, Canada. E-mail: wevans@cs.ubc.ca.

1 Introduction

Let S be a set of n points in the plane. A triangulation $T(S)$ of S is a maximal set of non-intersecting edges connecting points in S (that is, the addition of one more edge would create an intersection). The weight of an edge in $T(S)$ is the Euclidean distance between its endpoints. The weight of $T(S)$ is the sum of the weight of its edges. Computing the minimum weight triangulation $MWT(S)$ of a point set S is an old open problem. In fact, the complexity of this problem remains unresolved. It is not known to be NP-complete or polynomial-time solvable (see Garey and Johnson (1979)).

Recently, there has been a flurry of activity concerning this problem. This activity was sparked by a result of Keil (1994), who found the first non-trivial subgraph of the minimum weight triangulation of a point set. He showed that the $\sqrt{2}$-skeleton of a set of points S is a subgraph of the minimum weight triangulation of S. (β-skeletons, a proximity graph whose region of influence depends on the parameter β, were introduced by Kirkpatrick and Radke (1985) in the context of Computational Morphology.) The importance of this result rests on the fact that computing a subgraph of the minimum weight triangulation provides a polynomial time algorithm for computing the minimum weight triangulation, if the number of components in the subgraph is constant.

Gilbert (1979) and, independently, Klincsek (1980) provided an $O(n^3)$ time dynamic programming algorithm for computing the minimum weight triangulation of a simple polygon. This algorithm can be used to compute the minimum weight triangulation of a point set S in the following way. Suppose we are given a subgraph of the minimum weight triangulation that includes the convex hull $CH(S)$ of S and consists of k connected components. There are at most $\binom{n^2}{k-1}$ ways to choose $k-1$ edges to connect these components. For each choice that, along with the initial subgraph, forms a connected planar graph we use the algorithm of Gilbert or Klincsek to complete the triangulation optimally. Given a single component, each of the regions inside $CH(S)$ is a simply connected region and thus can be optimally triangulated using the algorithm of Gilbert or Klincsek. Therefore, in $n^{O(k)}$ time, $MWT(S)$ can be computed. (This time complexity can be reduced to $O(n^{k+2})$. See Cheng, Golin, and Tsang (1995).) Thus, the MWT problem is polynomial time solvable if a subgraph of the $MWT(S)$ consisting of a constant number of components can be computed in polynomial time.

After the original result by Keil (1994), other subgraphs of MWT were found by Yang and Xu (1994), and Cheng and Xu (1995). All of these subgraphs of the minimum weight triangulation, including Keil's, are also subgraphs of the light graph. Given a set S of n points in the plane, the light graph $L(S)$ of S is a set of edges connecting points in S with the following property. An edge $\overline{xy}$ $(x, y \in S)$ is in $L(S)$ provided that all segments $\overline{ab}$ with $a, b \in S \setminus \{x, y\}$ that properly intersect $\overline{xy}$ are strictly longer than $\overline{xy}$. The notion of a light graph and its relation to minimum weight triangulations was studied by Aichholzer et al. (1995). They showed that the weight of $MWT(S)$ is at least the weight of $L(S)$. Thus, if $L(S)$ is a triangulation, then it is the minimum weight triangulation. They also point out that there exist point sets such that not all edges in $L(S)$ are in $MWT(S)$. However, it was believed that light edges may help both in the exact computation of the minimum weight triangulation, since non-trivial subgraphs of $L(S)$ are contained in $MWT(S)$, and in approximate computations of the minimum weight triangulation. This belief was based on the observation that in practice, most light graphs seemed to be connected. In random point sets (uniformly distributed in the unit square) of up to 200 points, Aichholzer et al. observed that $L(S)$ was connected.

An alternate approach to computing the minimum weight triangulation has recently been proposed independently by Keil (1995) and Dickerson & Montague (1995). They both propose a method of computing a subgraph of the MWT by computing a *locally minimal skeleton* or LMT-skeleton for short. The LMT-skeleton of a set of points is the set of edges contained in every locally minimal triangulation. A triangulation is locally minimal if every edge of the triangulation is locally minimal. An edge e is locally minimal if one of the following holds.

(i) The edge e is on the convex hull.

(ii) The edge e is adjacent to two triangles which taken together form a non-convex quadrilateral.

(iii) The edge e is adjacent to two triangles that form a convex quadrilateral and e is shorter than the other diagonal of the quadrilateral.

Since the minimum weight triangulation is locally minimal, the LMT-skeleton is a subgraph of the MWT. Both conjecture that the LMT-skeleton is connected or only has a constant number of disconnected pieces. In fact Dickerson and Montague ran experiments on point sets up to 200 and found that the LMT-skeleton was connected.

In this paper, we show that there exist point sets such that the number of components in the light graph and LMT-skeleton is linear in the number of points. In fact, for n independent identically distributed points $X_1, \ldots, X_n$, drawn from the uniform distribution on $[0,1]^2$, we show that the expected number of components grows linearly in n. If C is the number of components, we also show that there exists a constant c_1 such that

$$\lim_{n\to\infty} \mathbf{P}\{C > c_1 n\} = 1.$$

These results suggest that the LMT-skeleton and the light graph do not help in resolving the time complexity of computing the minimum weight triangulation. The results are based on showing that a structure called a *diamond* causes both graphs to have isolated vertices and that the number of diamonds found in a random point set is high.

2 Diamonds and their properties

Given a set S of n points in the plane, a point $p \in S$ is called a *diamond* if two conditions are satisfied:

(i) The circle centered at p with radius $r = 1/\sqrt{n}$ contains no other data point;

(ii) The regular 18-gon centered at p with inscribed circle of radius r and aligned with the x-axis defines 18 quadrilateral sectors with vertices at neighboring tangential points, a vertex of the polygon, and the origin p (see figure below). Each of the sectors contains one and only one data point.

A *facet* of a diamond is one of the 18 regions sandwiched between the regular 18-gon and the inscribed circle of radius r centered at the diamond.

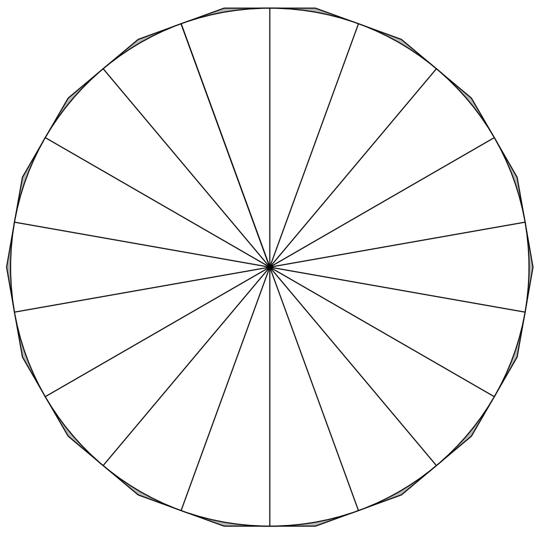

Figure 1: For the center to be a diamond, the inscribed circle must contain no other data point, and each facet must contain exactly one data point.

Let $f_0, f_1, ..., f_{17}$ be the points that lie within the facets of a diamond p in clockwise order about p. A diamond has the property that $\overline{f_{i-1}f_{i+1}}$ is shorter than r for all i (index sum modulo 18). This property implies the following:

Lemma 2.1 *If a set S of n points in the plane has a diamond , then the latter is an isolated vertex of $L(S)$.*

Proof: Let x be a point in $S \setminus \{p\}$. The edge $\overline{px}$ is longer than r and intersects some edge $\overline{f_{i-1}f_{i+1}}$ that, by construction, is shorter than r. ∎

A diamond is *secure* if no facet contains a point within angular distance $\gamma \equiv \arcsin(1/(2\cos(\pi/18))$ of another facet. A secure diamond has the property that every edge $\overline{pf_i}$ is shorter than $\overline{f_{i-2}f_{i+2}}$.

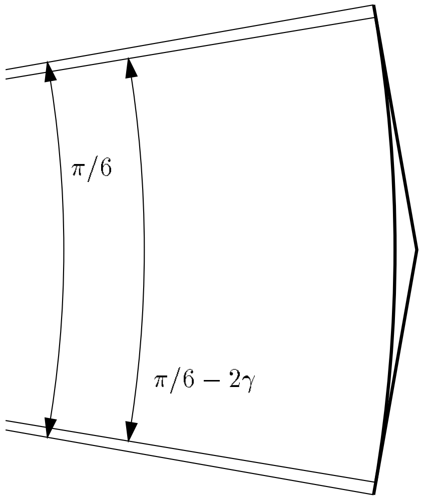

Figure 2: One of the 18 facets enlarged. A diamond is secure if the point within each facet is also within the $\pi/6 - 2\gamma$ sector.

Lemma 2.2 *If a set S of n points in the plane has a secure diamond , then the latter is an isolated vertex of the* LMT*-skeleton of S.*

Proof: We will show that there exists a locally minimal triangulation which does not include $\overline{px}$ for every $x \in S \setminus \{p\}$. If $x \in \{f_0, \ldots, f_{17}\}$ then without loss of generality assume $x = f_0$.

Consider the edges $T = \{\overline{pf_i},\ \overline{f_if_{i+2}} \mid i \text{ odd}\}$. The edges in T form nine empty triangles surrounding p. If the edges in T are locally minimal in every triangulation which contains them then there exists a locally minimal triangulation in which p is adjacent only to f_i for i odd. For example, the triangulation that contains T of minimum weight is one such locally minimal triangulation. Since x is not adjacent to p in this triangulation, $\overline{px}$ is not an edge in the LMT-skeleton of S.

To show that the edges in T are locally minimal in a triangulation we need only show that if an edge in T is the diagonal of a convex quadrilateral in the triangulation then the other diagonal is longer.

The edge $\overline{pf_i}$ is the diagonal of the quadrilateral $\{p, f_{i-2}, f_i, f_{i+2}\}$. Since p is a secure diamond, the edge $\overline{f_{i-2}f_{i+2}}$ is longer than $\overline{pf_i}$.

The edge $\overline{f_i f_{i+2}}$ may be the diagonal of a convex quadrilateral $\{p, f_i, y, f_{i+2}\}$ but y is outside the circle of radius r centered at p and thus $\overline{py}$ is longer than $\overline{f_i f_{i+2}}$. ■

Theorem 2.1 *For every $n \geq 19$, there is a set S of n points whose light graph and* LMT*-skeleton consist of at least $\lfloor n/19 \rfloor$ components.*

Proof: Follows from Lemmas 2.1 and 2.2. ■

Lemma 2.3 *If A, B, and C are the 18-gons associated with three distinct diamonds then $A \cap B \cap C = \emptyset$.*

Proof: Suppose $A \cap B \cap C$ is not empty. This implies that the circumcircles of two of these 18-gons must intersect so that one contains an arc of the other which is at least $\pi/3$ (and vice versa). This, in turn, implies that the inscribed circle of two of these 18-gons must intersect so that one contains an arc of the other which is at least $2\arccos(\sqrt{3}/(2\cos(\pi/18))) > 2\pi/9$. Thus an entire facet of one diamond is contained within the empty inscribed circle of another which contradicts the existence of a point within that facet. ■

3 Main result

Let D be the number of diamonds in the point set $S = \{X_i \mid 1 \leq i \leq n\}$ where each X_i is chosen uniformly at random in $[0,1]^2$. Lemma 2.1 implies that the number of components in $L(S)$ is at least D. Thus the following theorem shows that the expected number of components in $L(S)$ grows linearly in n.

Similarly, let D_s be the number of secure diamonds in the point set. Since Lemma 2.2 implies that the number of components in the LMT-skeleton of S is at least D_s, the next theorem shows that the expected number of components in the LMT-skeleton of S grows linearly in n.

Theorem 3.1 *There exists a positive constant ϵ such that $\mathbf{E}D \geq \epsilon n$ for all n. There exists a positive constant ϵ' such that $\mathbf{E}D_s \geq \epsilon' n$ for all n.*

Proof: Since the X_i's are i.i.d., we note that $\mathbf{E}D = n\mathbf{P}\{X_1 \text{ is a diamond}\}$.

$$\mathbf{P}\{X_1 \text{ is a diamond}|X_1\} \geq \begin{cases} 0 \text{ if } X_1 \text{ is within } 2r \text{ of the perimeter of } [0,1]^2 \\ \mathbf{P}\{N_0 = 0, N_1, \ldots, N_{18} = 1|X_1\} \text{ otherwise} \end{cases}$$

where $(N_0, N_1, \ldots, N_{18}, N_{19})$ is a multinomial random variable with parameters $n-1$ and $(\pi r^2, cr^2, \ldots, cr^2, 1-(\pi+18c)r^2)$, just as in a ball-in-20 urns experiment. The first urn is the circle of radius $r = 1/\sqrt{n}$ around X_1 and the next 18 urns are the 18 facets. The constant c is 1/18-th of the area difference between the circumscribing 18-gon and the circle, when the circle has unit radius. A bit of work shows that $c = \tan(\pi/18) - (\pi/18)$. We have

$$\begin{aligned} &\mathbf{P}\{N_0 = 0, N_1, \ldots, N_{18} = 1|X_1\} = \\ &\qquad \frac{(n-1)!}{(n-19)!}\left(cr^2\right)^{18}\left(1-(\pi+18c)r^2\right)^{n-19} \\ &\geq \ (n-18)^{18}c^{18}n^{-18}\left(1-(\pi+18c)/n\right)^n \\ &\sim \ c^{18}e^{-(\pi+18c)} \end{aligned}$$

as $n \to \infty$. Thus, the expected number of diamonds is at least

$$\begin{aligned} & nc^{18}e^{-(\pi+18c)}(1+o(1)) \\ & \quad \times \ \mathbf{P}\left\{X_1 \text{ not within } 2r \text{ of perimeter of } [0,1]^2\right\} \\ & \quad \geq \ nc^{18}e^{-(\pi+18c)}(1+o(1))(1-8r) \\ & \quad \sim \ nc^{18}e^{-(\pi+18c)} \ . \end{aligned}$$

The proof is almost identical for D_s. The only difference is that the constant c in the proof becomes somewhat smaller ($c = \tan(\pi/18) - (\pi/18) - (\tan(\gamma/18) - (\gamma/18))$). ■

REMARK. As $0 \leq D \leq n$, we have, for $0 < a < \epsilon$, with ϵ as in the Theorem above,

$$\mathbf{P}\{D \geq an\} \geq \frac{\mathbf{E}D - an}{n - an} \geq \frac{\epsilon - a}{1 - a} \ .$$

Take $a = (1-\delta)\epsilon$, $\delta > 0$, and conclude that $\mathbf{P}\{D \geq (1-\delta)n\epsilon\} > \delta\epsilon > 0$. Similarly, $\mathbf{P}\{D_s \geq (1-\delta)n\epsilon'\} > \delta\epsilon' > 0$ The concentration result from the next Theorem shows that these probabilities tend in fact to one with n.

Theorem 3.2 *If $\epsilon > 0$ is as in Theorem 3.1, then* $\lim_{n\to\infty} \mathbf{P}\{D \geq \epsilon n/2\} = 1$. *If $\epsilon' > 0$ is as in Theorem 3.1, then* $\lim_{n\to\infty} \mathbf{P}\{D_s \geq \epsilon' n/2\} = 1$.

Proof: The proof is a result of an exponential tail inequality due to McDiarmid (1989) whose origins can be traced back to Hoeffding (1963) and Azuma (1969) (see also Stout (1974), or Grimmett and Stirzaker (1992, p. 448)). If Z is an arbitrary function of n independent random variables $X_1, \ldots, X_n$, and if $Z(x_1, \ldots, x_n)$ changes by at most c when one of the x_i's changes value (with c not depending upon the x_j's), then

$$\mathbf{P}\{|Z - \mathbf{E}Z| > u\} \leq 2e^{-2u^2/(nc^2)} \ .$$

We will apply this inequality to D.

Let $D(i)$ be the number of diamonds in the set of $n-1$ points $S(i) = \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n\}$. We claim that

$$D(i) - 2 \leq D \leq D(i) + 2. \tag{1}$$

Consider the change in the number of diamonds caused by adding X_i to $S(i)$. In order to decrease the number of diamonds, X_i must fall within the 18-gons associated with one or more diamonds. At most two of these 18-gons can overlap any one point (Lemma 2.3). Thus we can destroy at most two diamonds with the addition of X_i which implies $D \geq D(i) - 2$. In order to increase the number of diamonds, X_i must either be a diamond itself or complete, i.e., fill in the last facet, of one or more diamonds. At most two such last facets can overlap (Lemma 2.3). Thus we can create at most two diamonds with the addition of X_i which implies $D \leq D(i) + 2$. If we replace one of the X_i's, D can thus change by at most four (apply (1) twice). By McDiarmid's inequality,

$$\mathbf{P}\{|D - \mathbf{E}D| \geq u\} \leq 2e^{-2u^2/(16n)} \ ,$$

where $u > 0$. By Theorem 3.1, $\mathbf{E}D \geq \epsilon n$. Thus,

$$\begin{aligned} \mathbf{P}\{D \leq \epsilon n/2\} & \leq \mathbf{P}\{D - \mathbf{E}D \leq -\epsilon n/2\} \\ & \leq 2e^{-n\epsilon^2/32} \\ & \to 0 \ . \end{aligned}$$

The proof is identical for D_s. ■

Theorems 3.1 and 3.2 above remain valid for the uniform distribution on any convex compact set of positive area. It is also applicable if on such a convex set, the data points are drawn from a density that remains bounded away from 0 and ∞.

4 Conclusion

These results imply that the light graph and the *LMT*-skeleton do not provide sufficient information in order to compute the minimum weight triangulation of a particular point set in polynomial time given the approach outlined in the introduction. In fact, for random point sets, our result only insists that the expected number of components is larger than one when n is larger than approximately 10^{51} which might explain why many isolated components were not observed in the experiements conducted on both the light graph and the *LMT*-skeleton.

5 References

O. Aichholzer, F. Aurenhammer, M. Taschwer and G. Rote, "Triangulations intersect nicely," *Proceedings of the 11th Annual ACM Symposium on Computational Geometry*, pp. 220–229, 1995.

N. Alon, J. Spencer, and P. Erdös, *The Probabilistic Method*, John Wiley, New York, 1992.

K. Azuma, "Weighted sums of certain dependent random variables," *Tohoku Mathematical Journal*, vol. 37, pp. 357–367, 1967.

S. Cheng, M. Golin, and J. Tsang, "Expected case analysis of β-skeletons with applications to the construction of minimum-weight triangulations," *Proceedings of the Seventh Canadian Conference on Computational Geometry*, 1995.

S. Cheng and Y. Xu, "Approaching the largest β-skeleton within a minimum weight triangulation," *to appear in Proceedings of the 12th Annual ACM Symposium on Computational Geometry*, 1996.

M.T. Dickerson and M.H. Montague, "The exact minimum weight triangulation," *to appear in Proceedings of the 12th Annual ACM Symposium on Computational Geometry*, 1996.

G. R. Grimmett and D. R. Stirzaker, *Probability and Random Processes*, Oxford University Press, 1992.

W. Hoeffding, "Probability inequalities for sums of bounded random variables," *Journal of the American Statistical Association*, vol. 58, pp. 13–30, 1963.

J.M. Keil, "Computing a subgraph of the minimum weight triangulation," *Computational Geometry: Theory and Applications*, vol. 4, pp. 13–26, 1994.

J.M. Keil, Personal communication, 1995.

D. Kirkpatrick and J. Radke, "A framework for computational morphology," *Computational Geometry (G. T. Toussaint, ed.)*, pp. 217–248, 1985.

G. Klincsek, "Minimal triangulations of polygonal domains," *Annals of Discrete Mathematics*, vol. 9, pp. 121–123, 1980.

C. McDiarmid, "On the method of bounded differences," in: *Surveys in Combinatorics 1989*, vol. 141, pp. 148–188, London Mathematical Society Lecture Notes Series, Cambridge University Press, Cambridge, 1989.

W. F. Stout, *Almost Sure Convergence*, Academic Press, New York, 1974.

B. Yang, Y. Xu, and Z. You, "A Chain Decomposition Algorithm for the Proof of a Property on Minimum Weight Triangulations," *Proceedings of the International Symposium on Algorithms and Complexity*, LNCS 834, Springer-Verlag, pp. 423–427, 1994.