Proximity Constraints and Representable Trees*
(extended abstract)

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Abstract. A family of proximity drawings of graphs called *open and closed β-drawings*, first defined in [16], and including the Gabriel, relative neighborhood and strip drawings, are investigated. Complete characterizations of which trees admit open β-drawings for $0 \leq \beta \leq \frac{1}{2sin^2(\pi/5)}$ and $\frac{1}{cos(2\pi/5)} < \beta < \infty$ or closed β-drawings for $0 \leq \beta < \frac{1}{2sin^2(\pi/5)}$ and $\frac{1}{cos(2\pi/5)} \leq \beta \leq \infty$ are given, as well as partial characterizations for other values of β. For $\beta < \infty$ in the intervals in which complete characterizations are given, it can be determined in linear time whether a tree admits an open or closed β-drawing, and, if so, such a drawing can be computed in linear time in the real RAM model. Finally, a complete characterization of all graphs which admit closed strip drawings is given.

1 Introduction and Overview

A drawing of a graph G maps the vertices of G to distinct points in the plane and each edge (u, v) of G to a simple curve between the points associated with u and v. Graph drawing algorithms and tools usually adopt given graphic standards. A widely used graphic standard represents all the edges as straight-line segments. Drawings within this standard are called *straight-line drawings*. A limited list of work on straight-line drawings includes [11–13, 15, 25]. Increasing attention has been recently given to *proximity drawings*. A survey on proximity drawings can be found in [6]. Given two points u and v of the plane, a *proximity region* of u and v is a suitably defined portion of the plane determined by u and v. A proximity drawing of G is a straight-line drawing such that: (i) for each edge (u, v) of G, the proximity region of the points representing u

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and v is empty (does not contain any other vertex); and (ii) for each pair of nonadjacent vertices u, v of G the proximity region of the points representing u and v is not empty. Several types of proximity regions have

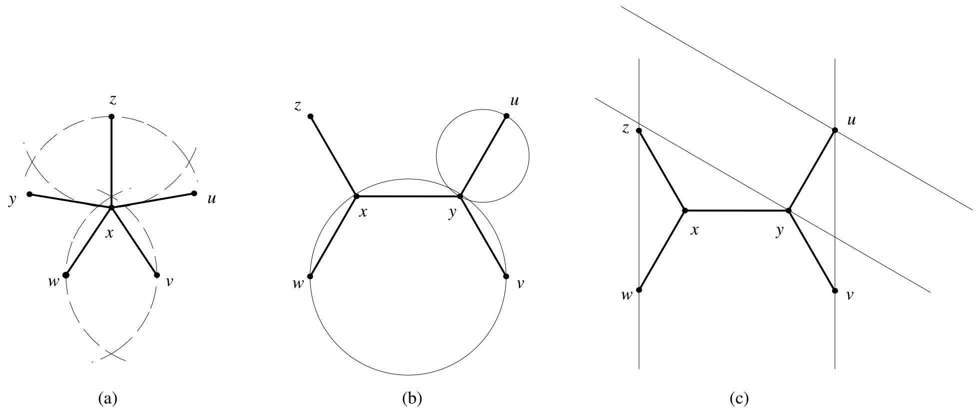

Fig. 1. Three proximity drawings.

been investigated, each one chosen for particular application purposes. Examples of proximity regions for a pair of points u, v include the *relative neighborhood region*: the intersection of the two open disks centered at u and at v and with the distance $d(u, v)$ as radius; the *Gabriel region*: the closed disk having u and v as antipodal points; and the *closed strip region*: the infinite closed strip having u and v on the boundary and width $d(u, v)$. For example, in Fig. 1(a) we show the proximity drawing of a tree T where the proximity regions are relative neighborhood regions. Observe that T contains edge (x, z) and the proximity region of the pair x, z is empty; conversely edge (w, v) is not in T and the proximity region of w, v contains x ($\{x, w, v\}$ were chosen to make angle $\angle wxv$ the smallest of the five angles). Tree T has no proximity drawing such that the proximity regions are Gabriel regions. Fig. 1(b) shows a proximity drawing of another tree T', using Gabriel regions; Fig. 1(c) shows a different proximity drawing of T', this time using proximity regions that are closed strips. Note that the drawings in Fig. 1(c) and Fig. 1(b) are the same even though the proximity regions are different.

In this paper we study the *proximity-drawability testing problem*, i.e., the problem of deciding whether a graph has a proximity drawing with a given type of proximity region. In particular we study the proximity-drawability of trees. We consider an infinite family of parametrized proximity regions, first introduced by [16], that covers the most well-known proximity regions presented in the literature. Due to space restrictions, most proofs have been omitted in this extended abstract. For full detailed proofs, we refer the reader to the technical report [3].

We consider two types of proximity regions:

Definition 1. Given a pair x, y of points in the plane, the *open β-region of influence of x and y*, and the *closed β-region of influence of x and y*, denoted by $R(x, y, \beta)$ and $R[x, y, \beta]$ respectively, are defined as follows:

1. For $0 < \beta < 1$, $R(x, y, \beta)$ is the intersection of the two open disks of radius $d(x, y)/(2\beta)$ passing through both x and y. $R[x, y, \beta]$ is the intersection of the two corresponding closed disks.
2. For $1 \leq \beta < \infty$, $R(x, y, \beta)$ is the intersection of the two open disks of radius $\beta d(x, y)/2$ and centered at the points $(1 - \beta/2)x + (\beta/2)y$ and $(\beta/2)x + (1 - \beta/2)y$. $R[x, y, \beta]$ is the intersection of the two corresponding closed disks.
3. $R(x, y, \infty)$ is the open infinite strip perpendicular to the line segment $\overline{xy}$ and $R[x, y, \infty]$ is the closed infinite strip perpendicular to the line segment $\overline{xy}$.
4. Finally, $R(x, y, 0)$ is the empty set and $R[x, y, 0]$ is the line segment connecting x and y.

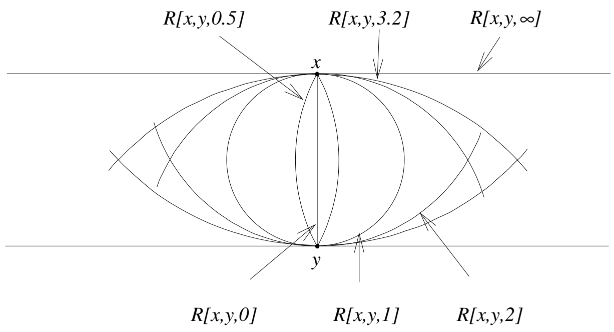

Fig. 2. A set of proximity regions $R[x, y, \beta]$

Fig. 2 illustrates some $[\beta]$-regions of a pair of points $\{x, y\}$ for several values of β. In Fig. 1, $R(x, z, 2)$, $R[u, y, 1]$, and $R[w, v, \infty]$ are examples of the relative neighborhood region, Gabriel region, and closed strip region, respectively.

1.1 Applications

The problem of testing whether a tree has a proximity drawing and, if so, of constructing such a drawing has applications in the area of graph drawing. Algorithms for straight-line drawings of trees are a classical field of investigation because of the number of practical situations in which the problem of representing a tree arises. For a small sample of papers that show algorithms for straight-line drawings of trees see [9, 8, 2]. Proximity drawings of trees have several interesting characteristics for visualization:

1. Neighbors of a given vertex cluster around that vertex;
2. The angles between consecutive edges are "large" (each angle is at least $\pi/3$); and
3. Proximity drawings of trees, as we will see later, have a relation to minimum spanning trees, another well studied class of tree-drawings [20, 10].

Note that the problem of constructing drawings with large angles (high-resolution drawings) has been studied in [18, 7]. For an up to date overview on graph drawing problems, applications, and algorithms, the reader is referred to [5].

Another application is concerned with pattern recognition. A classical way for associating a "shape" to a given distribution of points on the plane is to connect pairs of points that are deemed *close* by some proximity measure, computing in this way a graph, called a *proximity graph*, associated to the set of points. Many different measures of proximity have been defined (each giving rise to different types of proximity graphs) and among them the proximity regions described above play a central role [19],[22],[24],[14]. If, for example, one wishes to give a set of points the "shape" of a tree, it is necessary to determine which proximity regions will induce on the points such a shape. The results presented in this paper allow us to answer this type of question.

Finally, proximity drawing problems may be viewed as visibility problems: two points are mutually visible if a certain region between them contains no other point. From this point of view, the results in this paper deal with the problem of determining whether a tree can be realized as the visibility tree of a set of points.

1.2 Results

Let $\mathcal{T}(\beta)$ ($\mathcal{T}[\beta]$) be the class of trees that have a proximity drawing where the proximity region is the open (closed) β-region. We denote with $\mathcal{T}_k$ the set of all finite trees of maximum vertex degree at most k. Class $\mathcal{T}'$ is defined as the class of trees that have at least two vertices of degree three adjacent. The class $\overline{\mathcal{T}}$ are the so-called “forbidden” graphs defined in [2]. The results presented in this paper are listed below. Table 1 summarizes the characterization results and compares them with previous results, showing how the set of drawable trees changes as β changes. Columns of the table labelled “new” describe results of this paper; Columns labelled “previous” describe known results. A citation indicates that the result either first appeared in—or is a simple consequence of results appearing in—the cited papers.

- We give a complete characterization of proximity drawable trees with open regions for all β values such that $0 \leq \beta \leq \frac{1}{2\sin^2(\pi/5)} \simeq 1.45$ or such that $3.23 \simeq \frac{1}{\cos(2\pi/5)} < \beta < \infty$. Also, we give a complete characterization of proximity drawable trees with closed regions for all β values such that $0 \leq \beta < \frac{1}{2\sin^2(\pi/5)}$ or such that $\frac{1}{\cos(2\pi/5)} \leq \beta \leq \infty$. For all β values not in the above intervals, we give a partial characterization: we show that all trees in $\mathcal{T}_4$ and only trees in T_5 belong to $\mathcal{T}(\beta)$ and $\mathcal{T}[\beta]$.
- Based upon the characterization, for any β in the intervals mentioned above, we can decide in linear time whether a given tree belongs to $\mathcal{T}(\beta)$ or $\mathcal{T}[\beta]$.
- We describe linear time algorithms (in the Real-RAM model), which, given any β in the intervals mentioned above, and any tree $T \in \mathcal{T}(\beta)$ (or $\mathcal{T}[\beta]$), construct a proximity drawing of T with proximity region the open (or closed) β-region. Furthermore, we can produce in linear time such a proximity drawing for any tree in $\mathcal{T}_4$ and any value of β such that $1.45 < \beta < 3.23$.
- We show the relationships between proximity drawings discussed in this paper, Delaunay triangulations, and minimum spanning trees and exploit these relationships in our proofs.
- Furthermore, we show that the class of graphs that can be drawn with proximity region $R[x, y, \infty]$ consists of all binary forests. This is of particular interest since it is the only one of the proximity regions discussed in this paper which produces only acyclic graphs.

2 Preliminaries

We assume familiarity with the basic terminology of graph theory and computational geometry (see also [1], [21]). Let G be a graph. A *(β)-drawing* of G is a proximity drawing of G such that for each pair of points x, y the proximity region is $R(x, y, \beta)$. Analogously, a *$[\beta]$-drawing* of G is a proximity drawing of G such that for each pair of points x, y the proximity region is $R[x, y, \beta]$.

A graph is *(β)-drawable* if it has a (β)-drawing. Analogously, a graph is *$[\beta]$-drawable* if it has a $[\beta]$-drawing. (β)- and $[\beta]$-drawable graphs are also called (β)- and $[\beta]$-graphs, respectively.

A class of graphs is (β)-drawable (resp. $[\beta]$-drawable) if all its graphs are (β)-drawable (resp. $[\beta]$-drawable). A class of graphs is not (β)-drawable (resp. $[\beta]$-drawable) if it contains at least one graph that is not (β)-drawable (resp. $[\beta]$-drawable).

Given a set P of distinct points of the plane we denote by $G(P,\beta)$ the graph whose vertices correspond to the points of P and such that there is an edge (x,y) between two vertices corresponding to points x and y iff $R(x,y,\beta) \cap P = \emptyset$. It is easy to see that $G(P,\beta)$ has a (β)-drawing that is obtained by connecting with straight-line segments the points of P that correspond to adjacent vertices of $G(P,\beta)$. Hence, $G(P,\beta)$ is a (β)-graph. For simplifying the notation we denote, where this does not cause ambiguity, by $G(P,\beta)$ both the graph and its (β)-drawing and by P both the set of vertices and the points representing them in the drawing. Analogously, we denote by $G[P,\beta]$ the graph whose vertices corresond to the points of P and such that there is an edge between two vertices x and y iff $R[x,y,\beta] \cap (P - \{x,y\}) = \emptyset$.

A *delaunay triangulation* of P, denoted by $DT(P)$, is a planar graph whose vertices correspond to the points of P and whose edges are defined as follows. Construct a triangulation of P such that each interior triangle has the property that the open disk circumscribing the triangle contains no other point of P. The edges of $DT(P)$ are the edges of the triangles. A set P may admit more than one delaunay triangulation, but only if P contains four or more co-circular points. Obviously, the described triangulation of P is a planar straight-line drawing of $DT(P)$.

A *minimum spanning tree* of P, denoted by $MST(P)$, is a spanning tree of $DT(P)$ of minimum total edge length. In general, a set P may have many minimum spanning trees (for example, if P consists of the vertices of a regular polygon).

One of the quantities which is frequently used in analyzing (β)- and $[\beta]$-drawings is the angle $\alpha(\beta) = \inf\{\angle xzy | z \in R(x,y,\beta)\}$. Clearly if x and y are not adjacent in $G(P,\beta)$ $(G[P,\beta])$, then there is a point $z \in P$ such that $\angle xzy > \alpha(\beta)$ $(\angle xzy \geq \alpha(\beta))$. The converse, however, does not always hold. Note $\alpha(0) = \pi$ and $\alpha(\infty) = 0$. A related quantity is the angle $\gamma(\beta)$, which is defined, for $\beta \geq 2$, as follows. Consider the region $R(x,y,\beta)$ for two points x,y. Let z be a point on the boundary of $R(x,y,\beta)$ such that $d(x,y) = d(x,z)$. Then $\gamma(\beta) = \angle zxy$. Note that $\gamma(\infty) = \pi/2$, $\gamma(2) = \alpha(2) = \pi/3$, and, for $\beta > 2$, $\gamma(\beta) > \alpha(\beta)$.

By means of elementary geometric arguments the following property can be proved.

Property 1. β is related to $\alpha(\beta)$ and $\gamma(\beta)$ by the following equation:

$$\beta = \frac{1}{2\sin^2(\frac{\alpha(\beta)}{2})} = \frac{1}{\cos\gamma(\beta)}.$$

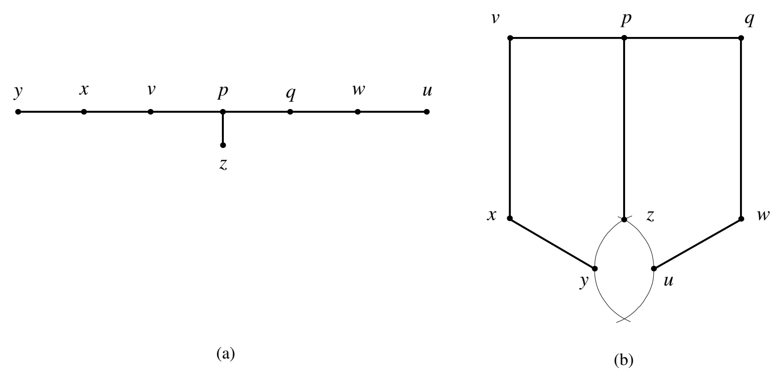

Fig. 3. (a) A [2]-stable and (b) non-[2]-stable drawing of the same tree.

Finally, an induced subgraph of a graph G which is obtained by repeated removal of leaves is called a *pruning* of G. Let G be a graph which admits a (β)-drawing ($[\beta]$-drawing) Γ and let G' be a pruning of G obtained by removing the set of vertices V'. Let Γ' be obtained from Γ by removing the points corresponding to the set V'. If for all prunings G' of G, Γ' is a (β)-drawing of G', then G is called a *(β)-stable* (*$[\beta]$-stable*) graph and Γ is a *(β)-stable* (*$[\beta]$-stable*) drawing of G. Observe that if Γ is a (β)-stable (or $[\beta]$-stable) drawing of a tree T, then for any pair of non-adjacent vertices x and y in T, there is a vertex v on the (unique) path between x and y such that v is contained in the proximity region of x and y. Fig. 3(a) and 3(b) show stable and non-stable $[\beta]$-drawings, respectively, of the same tree for $\beta = 2$.

3 Points, Graphs, and Drawings

Here we study the relations between (β)- and $[\beta]$-graphs, and we relate (β)- and $[\beta]$-graphs to minimum spanning trees and delaunay triangulations. In the following P denotes a finite set of points of the plane.

3.1 Properties of (β)- and $[\beta]$-graphs

Property 2. If β_1 and β_2 are such that $0 \leq \beta_1 < \beta_2 \leq \infty$ then

$$G[P,\beta_2] \subseteq G(P,\beta_2) \subseteq G[P,\beta_1] \subseteq G(P,\beta_1).$$

Property 2 has the following consequences.

Property 3. For a given P the number of edges of $G(P,\beta)$ and $G[P,\beta]$ is a non-increasing function of β.

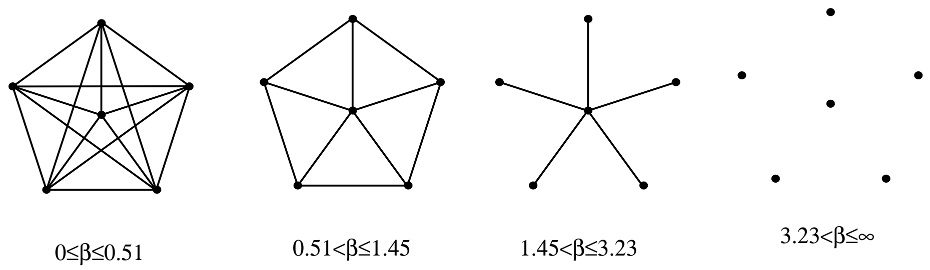

Fig. 4. Different $G(P,\beta)$ as β varies.

Fig. 4 shows a set of points P and the different graphs $G(P,\beta)$ as β ranges from 0 to ∞.

Property 4. For $\beta > 1$, $G(P,\beta)$ and $G[P,\beta]$ are planar. Also, $G[P,1]$ is planar. For $\beta < 1$, $G(P,\beta)$ and $G[P,\beta]$ are not necessarily planar. Also, $G(P,1)$ is not necessarily planar

Using the fact, proven in [23], that $G(P,2)$ is connected, the following property can be established.

Property 5. For $\beta < 2$, $G(P,\beta)$ and $G[P,\beta]$ are connected. Also, $G(P,2)$ is connected. For $\beta > 2$, $G(P,\beta)$ and $G[P,\beta]$ are not necessarily connected. Also, $G[P,2]$ is not necessarily connected.

Properties 2 and 5 along with the fact that $G(P,\beta)$ is a tree imply the following lemma.

Lemma 1. *Let T be a tree, let β such that $0 \le \beta < 2$ and let P be a set of points such that $G(P,\beta)$ is a (β)-drawing of T. Then,*

1. *For every β' such that $\beta \le \beta' \le 2$, $G(P,\beta')$ is a (β')-drawing of T, and*

2. *For every β' such that $\beta \leq \beta' < 2$, $G[P, \beta']$ is a $[\beta']$-drawing of T.*

Given a set of points P, a value of β is *P-critical* when for each $\epsilon > 0$ we have that $G(P, \beta+\epsilon) \subset G(P, \beta)$. Roughly speaking a value is P-critical when in that value $G(P, \beta)$ "loses" at least one edge. From Property 3 and because P has a finite number of points we have that, for a given P, there are only finitely many P-critical values of β.

Lemma 2. *Given a P, a value β is P-critical if and only if $G(P, \beta) \neq G[P, \beta]$. Also, let $C = \{\beta_i, 0 \leq \beta_1 < \beta_2 < \ldots < \beta_k \leq \infty\}$ the set of all the P-critical values of β:*

1. *For all $\beta \in (0, \beta_1)$, $G[P, 0] = G[P, \beta] = G(P, \beta) = G(P, \beta_1)$.*
2. *For each $i < k$, for all $\beta \in (\beta_i, \beta_{i+1})$, $G[P, \beta_i] = G[P, \beta] = G(P, \beta) = G(P, \beta_{i+1})$.*
3. *For all $\beta \in (\beta_k, \infty)$, $G[P, \beta_k] = G[P, \beta] = G(P, \beta) = G(P, \infty)$.*

Lemma 2 yields an equivalent definition of P-critical: a value of β is P-critical when for each $\epsilon > 0$ we have that $G[P, \beta] \subset G[P, \beta - \epsilon]$.

3.2 (β)- and $[\beta]$-graphs and Minimum Spanning Trees

In this subsection, we exhibit the close relationship of the minimum spanning tree of a set of points with (β)- and $[\beta]$-graphs.

Theorem 1.

1. *If* $0 \leq \beta < 2$, *then* $MST(P) \subseteq G(P,\beta)$ *and* $MST(P) \subseteq G[P,\beta]$.
2. $MST(P) \subseteq G(P,2)$.
3. *There exists a* P *such that* $G[P,2]$ *is a tree but* $MST(P) \neq G[P,2]$.
4. *If* $2 < \beta < \infty$, *then there exists a* P *such that* $G[P,\beta] = G(P,\beta)$ *is a tree but* $MST(P) \neq G[P,\beta]$.
5. *There exists a* P *such that* $G(P,\infty)$ *is a tree but* $MST(P) \neq G(P,\infty)$.

The relationship between $G[P,\infty]$ and $MST(P)$ is discussed in Section 5. Furthermore, the following property is proved in [16].

Lemma 3. *For each edge* $e \in G(P,\infty)$, *there exists a minimum spanning tree of* P *containing* e.

3.3 (β)- and [β]-graphs and Delaunay Triangulations

In this subsection, we exhibit the close relationship of the Delaunay triangulation of a set of points with (β)- and [β]-graphs.

Lemma 4. *For* $\beta > 1$, $G(P,\beta)$ *and* $G[P,\beta]$ *are subgraphs of* $DT(P)$. *Also,* $G[P,1]$ *is a subgraph of* $DT(P)$.

From Property 4 and from the planarity of $DT(P)$ it follows that for $\beta < 1$ the above property, in general, does not hold.

Lemma 5 and Theorem 2 generalize analogous results that have been given in [2] for $\beta = 1$ and can be proved with similar techniques.

Lemma 5. *Given a set* P, *consider a value* β *such that* $0 \leq \beta \leq 1$. *Let* (u,v) *be an edge of* $DT(P)$.

1. *If* (u,v) *is contained in two triangles* $\triangle(uvc)$ *and* $\triangle(uvd)$ *of* $DT(P)$ *then*
   (a) $(u,v) \in G[P,\beta]$ *iff* $\angle ucv$ *and* $\angle udv$ *are both less than* $\alpha(\beta)$;
   (b) $(u,v) \in G(P,\beta)$ *iff* $\angle ucv$ *and* $\angle udv$ *are both less than or equal to* $\alpha(\beta)$.
2. *If* (u,v) *is contained in only one triangle* $\triangle(uvc)$ *of* $DT(P)$ *then*
   (a) $(u,v) \in G[P,\beta]$ *iff* $\angle ucv$ *is less than* $\alpha(\beta)$;
   (b) $(u,v) \in G(P,\beta)$ *iff* $\angle ucv$ *is less than or equal to* $\alpha(\beta)$.

The following theorem characterizes the relationship between $G(P,\beta)$, $G[P,\beta]$, and delaunay triangulations, when (1) $0 < \beta \leq 1$ and (2) $G(P,\beta)$ and $G[P,\beta]$ are trees.

Theorem 2. *Given a P, consider a value β such that $0 \leq \beta \leq 1$. We have that if $G[P, \beta]$ ($G(P, \beta)$) is a tree, then for each cycle C of $DT(P)$ there exists an edge $(u, v) \in C$ not in $G[P, \beta]$ ($G(P, \beta)$) such that for some point $p \in P$, $\triangle(upv)$ is a face of $DT(P)$, $\triangle(upv)$ lies inside C, and $\angle upv \geq \alpha(\beta)$ ($\angle apb > \alpha(\beta)$).*

4 Classes of Trees

We start by showing that trees with vertices of degree greater than or equal to 6 are neither (β)- nor $[\beta]$-drawable.

Lemma 6. *Let $G(P, \beta)$ (resp. $G[P, \beta]$) be a tree. If $0 \leq \beta \leq 2$, the angle between any two consecutive edges of $G(P, \beta)$ (resp. $G[P, \beta]$) is greater than $\alpha(\beta)$; if $2 < \beta \leq \infty$, the angle is at least $\gamma(\beta)$.*

From Property 1 it follows that for $0 \leq \beta \leq 2$, $\alpha(\beta) \geq \pi/3$ and for $2 \leq \beta \leq \infty$, $\gamma(\beta) > \pi/3$. Hence, lemma 6 allows us to prove the following theorem.

Theorem 3. *(β)-trees and $[\beta]$-trees have no vertices of degree more than 5.*

Because of the above theorem we restrict our attention to the drawability of classes $\mathcal{T}_k$ with $k < 6$. We begin with a characterization of the class $\mathcal{T}_2$.

Theorem 4. *Every tree in class $\mathcal{T}_2$ admits a (β)-stable drawing and a $[\beta']$-stable drawing for all values of $\beta, \beta' \neq 0$. Class $\mathcal{T}_2$ is not (0)-drawable.*

We characterize the class of trees $\mathcal{T}_3$ (see [3] for the drawing algorithm).

Theorem 5. *Every tree in class $\mathcal{T}_3$ admits a (β)-stable drawing and a $[\beta']$-stable drawing for all values of β, β' such that $\frac{2}{3} < \beta, \beta' \leq \infty$. Furthermore, given a $T \in \mathcal{T}_3$ and a β such that $\frac{2}{3} < \beta \leq \infty$, a (β)-drawing and a $[\beta]$-drawing of T can be computed in linear time in the real RAM model. Class $\mathcal{T}_3$ is neither (β)-drawable nor $[\beta]$-drawable for all values of β such that $0 \leq \beta \leq \frac{2}{3}$*

When $\beta = \frac{2}{3}$, we have the following.

Lemma 7. *A tree $T \in \mathcal{T}_3$ is $[\frac{2}{3}]$-drawable if and only if T has no two adjacent vertices of degree 3.*

For $\beta = \infty$, we note that the class $\mathcal{T}_3$ is the only class of trees that admit a $[\infty]$-drawing.

Lemma 8. *A tree T has a $[\infty]$-drawing if and only if $T \in \mathcal{T}_3$.*

The following is a characterization of the class $\mathcal{T}_4$ (refer to [3] for a description of the drawing algorithm).

Theorem 6. *Every tree in class $\mathcal{T}_4$ admits a (β)-stable drawing and a $[\beta']$-stable drawing for all values of β, β' such that $1 < \beta, \beta' < \infty$. Furthermore, given a $T \in \mathcal{T}_4$ and a β such that $1 < \beta < \infty$, a (β)-drawing and a $[\beta]$-drawing of T can be computed in linear time in the real RAM model. Class $\mathcal{T}_4$ is neither (β)-drawable nor $[\beta]$-drawable for any other values of β.*

The range of values of β in which $\mathcal{T}_5$ is (β)- or $[\beta]$-drawable is as yet unknown. In [2], it is shown that $\mathcal{T}_5$ is both (2)- and [2]-drawable. As a consequence of Lemma 6, we see that no trees having any vertices of degree 5 can be (β)- or $[\beta]$-drawn for any $\beta < \frac{1}{2\sin^2(\pi/5)}$ or for any $\beta > \frac{1}{\cos(2\pi/5)}$.

We conclude this section with our main result.

Theorem 7. *As β ranges from 0 to ∞, the sets $\mathcal{T}(\beta)$ and $\mathcal{T}[\beta]$ change as shown in Table 1.*

5 A Characterization of $G[P, \infty]$

Lemma 9. *For each finite set of points of the plane P, $G[P, \infty]$ is a subgraph of the intersection of all minimum spanning trees of P.*

However, the converse of Lemma 9 does not hold, as can be seen by choosing P to be the corners of a square. An interesting consequence of Lemma 9 is the following:

Lemma 10. *Every $[\infty]$-drawable tree is $[\infty]$-stable.*

Theorem 8. *A graph G is $[\infty]$-drawable if and only if every connected component of G is in $\mathcal{T}_3$ and G is not one of the following graphs: two non-adjacent vertices, a vertex and a non-adjacent edge, or a pair of non-adjacent edges.*

6 Directions for Further Research

Question 1. For $\beta \neq 2$ such that $\frac{1}{2sin^2(\pi/5)} \leq \beta \leq \frac{1}{cos(2\pi/5)}$, which trees admit open or closed β-drawings?

Question 2. Does every tree which admits a β-drawing admit a β-stable drawing?

Question 3. Which graphs have (∞)-drawings?

To date, little work has been done on the problems of characterizing other families of β-drawable graphs. In [17], Lubiw and Sleumer showed that maximal outer-planar graphs admit both [1]-drawings (Gabriel drawings) and (2)-drawings (relative neighborhood drawings). It would be particularly interesting to determine which triangulated planar graphs were β-drawable for different values of β.

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	β	$\mathcal{T}(\beta)$ previous	$\mathcal{T}[\beta]$ previous	$\mathcal{T}(\beta)$ new	$\mathcal{T}[\beta]$ new
1	$\beta = 0$	—	—	$\mathcal{T}(\beta) = \{K_1, K_2\}$	$\mathcal{T}[\beta] = \mathcal{T}_2$
2	$0 < \beta < \frac{2}{3}$	—	—	$\mathcal{T}(\beta) = \mathcal{T}_2$	$\mathcal{T}[\beta] = \mathcal{T}_2$
3	$\beta = \frac{2}{3}$	—	—	$\mathcal{T}(\beta) = \mathcal{T}_2$	$\mathcal{T}[\beta] = \mathcal{T}_3 - T'$
4	$\frac{2}{3} < \beta < 1$	—	—	$\mathcal{T}(\beta) = \mathcal{T}_3$	$\mathcal{T}[\beta] = \mathcal{T}_3$
5	$\beta = 1$	$\mathcal{T}(\beta) = \mathcal{T}_3$ [2]	$\mathcal{T}[\beta] = \mathcal{T}_4 - \overline{\mathcal{T}}$ [2]	—	—
6	$1 < \beta < \frac{1}{2\sin^2(\pi/5)}$	$\mathcal{T}_3 \subseteq \mathcal{T}(\beta)$ [4, 16]	$\mathcal{T}_3 \subseteq \mathcal{T}[\beta]$ [26, 16]	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}[\beta] = T_4$
7	$\beta = \frac{1}{2\sin^2(\pi/5)}$	$\mathcal{T}_3 \subseteq \mathcal{T}(\beta)$ [4, 16]	$\mathcal{T}_3 \subseteq \mathcal{T}[\beta]$ [26, 16]	$\mathcal{T}(\beta) = \mathcal{T}_4$	$\mathcal{T}_4 \subset T[\beta] \subset \mathcal{T}_5$
8	$\frac{1}{2\sin^2(\pi/5)} < \beta < 2$	$\mathcal{T}_3 \subseteq \mathcal{T}(\beta)$ [4, 16]	$\mathcal{T}_3 \subseteq \mathcal{T}[\beta]$ [26, 16]	$T_4 \subset \mathcal{T}(\beta) \subseteq \mathcal{T}_5$	$\mathcal{T}_4 \subset \mathcal{T}[\beta] \subseteq \mathcal{T}_5$
9	$\beta = 2$	$\mathcal{T}(\beta) = \mathcal{T}_5$ [2]	$\mathcal{T}[\beta] = T_5$ [2]	—	—
10	$2 < \beta < \frac{1}{\cos(2\pi/5)}$	—	—	$T_4 \subset \mathcal{T}(\beta) \subseteq T_5$	$T_4 \subset T[\beta] \subseteq T_5$
11	$\beta = \frac{1}{\cos(2\pi/5)}$	—	—	$T_4 \subset \mathcal{T}(\beta) \subset T_5$	$T[\beta] = T_4$
12	$\frac{1}{\cos(2\pi/5)} < \beta < \infty$	—	—	$\mathcal{T}(\beta) = T_4$	$T[\beta] = T_4$
13	$\beta = \infty$	—	—	$T_3 \subset \mathcal{T}(\beta) \subset T_4$	$T[\beta] = T_3$

Table 1. Summarizing the characterization results