Ordered Theta Graphs[†]

Prosenjit Bose[‡] Joachim Gudmundsson[§] Pat Morin[‡]

Abstract

Let V be a set of n points in $\mathbb{R}^2$. The θ-graph of V is a geometric graph with vertex set V that has been studied extensively and which has several nice properties. We introduce a new variant of θ-graphs which we call *ordered θ-graphs*. These are graphs that are built incrementally by inserting the vertices one by one so that the resulting graph depends on the insertion order. We show that careful insertion orders can produce graphs with desirable properties.

1 Introduction

Let V be a set of n points in the plane. A *t-spanner* of V is a geometric graph $G = (V, E)$ whose edges are weighted by the distance between their endpoints and which has the property

$$\max\left\{\frac{\|uv\|_G}{\|uv\|} : u, v \in V, u \neq v\right\} \leq t \enspace ,$$

where $\|uv\|$, respectively $\|uv\|_G$, denotes the Euclidean distance, respectively the length of the shortest path in G, between u and v. We call a path P from u to v a *t-path* if $\|uv\|_P/\|uv\| \leq t$. Thus, G is a t-spanner if and only if every pair of vertices in G has a t-path between them.

It has long been known that, for any constant $t > 1$, every point set V has a t-spanner with $O(n)$ edges. One such construction is the θ-graph of V [6, 7]. Let $\theta < \pi/4$ be a value such that $k_\theta = 2\pi/\theta$ is a positive integer. The θ-graph of V is obtained by drawing $k_\theta = 2\pi/\theta$ non-overlapping cones around each $p \in S$, each spanning an angle of θ, and connecting p to the point in each cone whose orthogonal projection onto one of the walls of the cone is closest to p. The result is a t_θ-spanner with at most nk_θ edges. Here, and throughout the rest of the paper,

$$t_\theta = 1/(\cos(\theta) - \sin(\theta)) \enspace .$$

This paper studies a variant of θ-graphs that we call *ordered θ-graphs*. An ordered θ-graph of V is obtained by inserting the points of V in some order. When a point p is inserted, we draw the same cones around p and connect p to its closest previously-inserted neighbour in each cone. An ordered θ-graph is dependent on the order imposed on V; different orderings of V can produce different graphs. Nevertheless, in Section 2 we show that ordered θ-graphs are also t_θ-spanners, regardless of the ordering used.

We also study different properties that can be obtained by carefully choosing orderings of V. In Section 3 we show that every point set has an ordering such that the ordered θ-graph has maximum degree $O(k_\theta \log n)$. In Section 4 we show that for every point set there exists an ordering such that, in the resulting ordered θ-graph, there is a t_θ-path with $O(\log n)$ edges between every pair of vertices. We say that such a graph has $O(\log n)$ *spanner diameter*.

The two results described above are not new. Sink spanners [1] are a transformation of θ-graphs that achieve a somewhat stronger result, namely a t_θ-spanner with degree at most ${k_\theta}^2 + k_\theta$. Skiplist spanners [2] use random-sampling of vertices to obtain a graph with spanner diameter $O(\log n)$, and $O(nk_\theta)$ edges. However, ordered θ-graphs show the existence of spanners with small degree and spanners with small diameter using a unified approach. Also, in the case of small diameter, the proof improves the constants in skiplist spanners since an ordered θ-graphs of n points contains at most nk_θ edges.

In Section 5 we show that two generalizations of θ-graphs also apply to ordered θ-graphs. Finally, in Section 6 we summarize and conclude with open problems.

2 Ordered θ-Graphs

In this section, we give a formal definition of ordered θ-graphs and prove some basic properties about them. Let V be a set of n points in the plane and let $\theta < \pi/4$ be an angle such that $k_\theta = 2\pi/\theta$ is a positive integer. Define any total order π on V so that π_v is the rank of v in this order. Let P_v denote the predecessors of v in π, i.e., $P_v = \{u \in V : \pi_u < \pi_v\}$ and let S_v denote the successors of v in π, i.e., $S_v = V \setminus (\{v\} \cup P_v)$.

The π-ordered θ-graph of V is obtained by repeating the following for each point $v \in V$ (see Figure 1). Partition the plane into k_θ cones each spanning an angle of θ

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[‡]School of Computer Science, Carleton University, {jit,morin}@cs.carleton.ca

[§]Department of Information and Computing Sciences, Utrecht University, joachim@cs.uu.nl

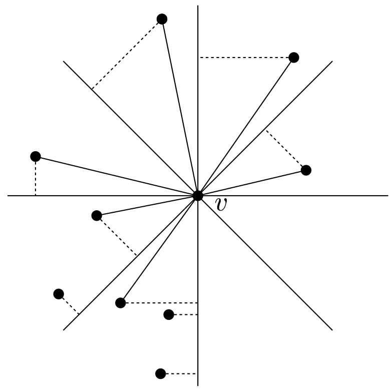

Figure 1: The θ-cones of v.

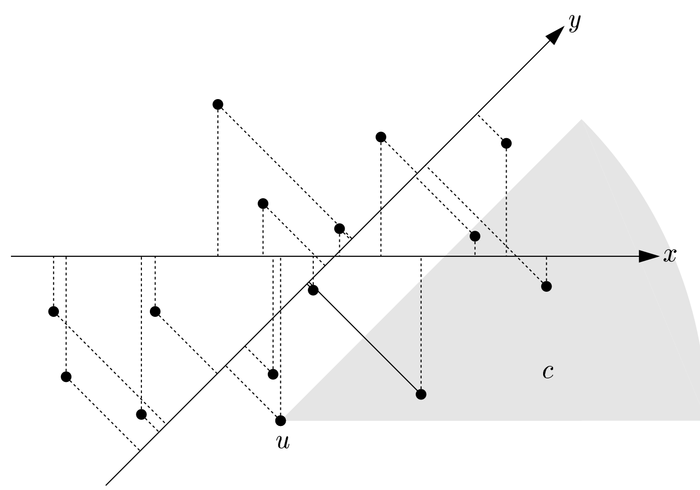

Figure 2: Finding the neighbour of u in c is equivalent to finding the point in the upper right quadrant of u with minimum y coordinate.

and having their apexes on v. Next, project each point of P_v orthogonally onto the counterclockwise wall of the cone that contains it. Finally, make an edge joining v to the point in each cone whose projection is closest to v.

Lemma 1. *For any point set V and any ordering π, the π-ordered θ-graph $G = (V, E)$ of V is a t_θ-spanner with at most $k_\theta n$ edges.*

Proof. It follows immediately from the definition that G has at most $k_\theta n$ edges.

To prove that G is a t_θ-spanner, consider any pair of points $u, v \in V$. We use induction on $\max\{\pi_u, \pi_v\}$. Without loss of generality, assume $\pi_u > \pi_v$. If $\pi_u = 2$ then $\pi_v = 1$ and there is a direct edge from u to v so the claim is trivial. Otherwise, consider the θ-cone c of u that contains v and let w be the neighbour of u in c. If $w = v$ then we are done. Otherwise, the projection of w onto the counterclockwise wall of c is closer than the projection of v onto the counterclockwise wall of c. Simple trigonometry shows that

$$\|uw\| + t_\theta \|wv\| \le t_\theta \|uv\| \ .$$

Since $\pi_w < \pi_u$ the inductive hypothesis states that $\|wv\|_G \le t_\theta \|wv\|$, which proves the claim. □

Note that the above proof gives an algorithm for finding a path between u and v that works by constructing the path from both ends. If $\pi_u > \pi_v$ then the second vertex in the path from u to v is to the neighbour w of u that is contained in the same θ-cone of u as v. Otherwise, the second last vertex in the path from u to v is the neighbour w of v that is contained in the same θ-cone of v as u. We call the path produced by this algorithm the *θ-path* from u to v.

While it is nice to know that ordered θ-graphs have good spanning properties, it is still important to be able to compute them efficiently.

Lemma 2. *For any set V of n points in $\mathbb{R}^2$ and any ordering π on V, the π-ordered θ-graph of V can be computed in $O(k_\theta n \log n)$ time.*

Proof. The crucial part of the construction algorithm is finding the neighbours of each point $v \in S$. For this, we use k_θ range trees [3], one for each cone direction. In each tree we store the points of V with their coordinates represented in terms of the walls of one of the cones, i.e., in a coordinate system in which the x and y axes meet at an angle of θ (see Figure 2).

Once this coordinate transformation is done, finding the neighbour of u in a cone c is equivalent to finding the point with minimum y coordinate that has both x and y coordinates larger than the x and y coordinates of u. These *dominance queries* are exactly the types of queries that are answered by range trees.

To construct the π-ordered θ-graph we work backwards through the sequence π. We first choose the point $v \in V$ such that $\pi_v = n$, delete v from all range trees and then use the range trees to find the neighbours of v. We continue in this manner for the point $u \in V$ such that $\pi_u = n - 1$ and so on until we have computed the entire π-ordered θ-graph of V.

Thus, computing the neighbours of each point v involves k_θ deletions and searches in range trees. A version of range trees that supports construction in $O(n \log n)$ time, and queries and deletions in $O(\log n)$ amortized time is given by Mehlhorn and Näher [9]. Using this implementation of range trees, the above algorithm constructs the π-ordered θ-graph in $O(k_\theta n \log n)$ time. □

3 Ordered θ-Graphs with Low Degree

So far, we have established that ordered θ-graphs have the same spanning properties as θ-graphs and can be constructed efficiently. In this section, we show that a carefully chosen ordering π yields a π-ordered θ-graph in which each vertex has small degree.

Theorem 1. *Every point set V of size n has an ordered θ-graph in which each vertex has degree at most $k_\theta(H_{n-1}+1)$.*[1] *Furthermore, this ordered θ-graph can be constructed in $O(k_\theta n \log n)$ time.*

Proof. We construct the ordering incrementally. Initally we choose an arbitrary vertex $v_n \in V$, mark v_n and set $\pi_{v_n} \leftarrow n$. This determines up to k_θ edges of the π-ordered θ-graph. In the second step we choose some unmarked vertex $v_{n-1} \in V$ of degree 1, mark v_{n-1} and set $\pi_{v_{n-1}} \leftarrow n-1$. In general, in the ith step (beginning at $i=0$) we choose an unmarked vertex v of maximum degree, mark v and set $\pi_v \leftarrow n-i$, thereby fixing up to k_θ more edges of π-ordered θ-graph.

To prove that the above algorithm gives the desired degree bound, we relate it to the following pebble game: Imagine we have a set of n buckets and two players P_1 and P_2. In one round, P_1 removes a bucket containing the maximum number of balls and P_2 adds a total of at most k_θ balls to some subset of buckets. The game ends after n rounds. Dietz and Sleator [4] show that, no matter what P_2's strategy is, the maximum number of balls contained in any bucket at any point during the execution of the game does not exceed $k_\theta(H_{n-1}+1)$. This game and the above algorithm for constructing π are completely analagous if we think of the buckets as V, P_1 as our algorithm and P_2 as the geometry of V that determines which edges that are fixed each time we fix the rank of a vertex in π. Thus, the result of Dietz and Sleator implies that the degree of a vertex in the resulting π-ordered θ-graph does not exceed $k_\theta(H_{n-1}+1)$, as required.

As in the proof of Lemma 2, the above algorithm is easily implemented to run in $O(k_\theta n \log n)$ time using the deletion only range tree data structure of Mehlhorn and Näher [9]. □

While the bound in the proof of Theorem 1 is optimal for the pebble game used in the proof, we have no example of a point set for which every ordering produces an ordered θ-graph with $\omega(k_\theta)$ degree at some vertex.

Before continuing, we remark that the algorithm in the proof of Theorem 1 can produce a graph with diameter $n-1$. This happens, for example if the point set V lies on a line and the algorithm begins by choosing one of the extreme points of V.

[1] Here, and throughout, $H_m = \sum_{i=1}^{m} 1/i$ denotes the mth harmonic number. It is well known that $\ln m \le H_m \le \ln n + 1$ [5].

4 Ordered θ-Graphs with Logarithmic Diameter

In this section, we show that an ordered θ-graph constructed by choosing a random ordering has θ-paths that use only $O(\log n)$ edges.

Theorem 2. *Let $G=(V,E)$ be an ordered θ-graph of V obtained by taking the points of V in random order. Then the probability that there exists a pair $u,v \in V$ such that the θ-path from u to v has more than $c\log n$ edges is $n^{-\Omega(c)}$.*

Proof. Consider two consecutive steps of the algorithm for finding a θ-path from u to v. These steps either complete the path, or reduce the problem of finding a path between $u,v \in V$ to a problem of finding a path between $w,x \in V$. We say that two consecutive steps are *successful* if they complete the path, or if $\max\{\pi_w,\pi_x\} < \alpha \max\{\pi_u,\pi_v\}$, for some constant $0<\alpha<1$. A simple cases analysis shows that the probability that two consecutive steps are successful is at least α^2, and this statement is true regardless of any previous steps taken by the algorithm.

Observe that, since the length of a path is bounded by n, any run of the algorithm for finding a θ-path has at most $\log_{1/\alpha^2} n$ successes. Therefore, if we let X denote the number of edges in the θ-path from u to v and let B denote a binomial$(2c\log_{1/\alpha^2} n, \alpha^2)$ random variable then

$$\begin{aligned}\Pr\left\{X \ge 2c\log_{1/\alpha^2} n\right\} &\le \Pr\left\{B \le 2\log 1/\alpha^2 n\right\} \\ &= \Pr\left\{B \le \frac{1}{\alpha^2 c}\mathbf{E}B\right\} \\ &\le n^{-\left(1-\frac{1}{\alpha^2 c}\right)^2\alpha^2 c/\ln(1/\alpha^2)} \\ &= n^{-\Omega(c)} \ ,\end{aligned}$$

where the second inequality follows from Chernoff's bound on the head of the binomial distribution. Thus, the probability that there exists any pair of vertices $u,v \in V$ such that the θ-path from u to v has more than $2c\log_{1/\alpha^2} n$ edges is at most $\binom{n}{2}n^{-\Omega(c)} = n^{-\Omega(c)}$, as required. □

We remark that, unfortunately, a random ordering does not necessarily produce an ordered θ-graph in which every vertex has low degree. For example, consider a point set V such that the (unordered) θ-graph of V has a vertex v of degree $n-1$. In this case, the expected degree of v in the randomly ordered θ-graph is

$$\frac{1}{n}\sum_{i=1}^{n}(n-i) = (n-1)/2 \ .$$

5 Generalizations

In this section we discuss two generalizations of ordered θ-graphs that follow from the corresponding generalizations of (unordered) θ-graphs.

5.1 Higher Dimensions

Ruppert and Seidel [10] give a natural generalization of θ-graphs to d-dimensions that can be constructed in $O(n \log^{d-1} n)$ time and yield a t_θ-spanner with $O(k_{d,\theta} n)$ edges, where $k_{d,\theta} = (d/\theta)^{O(d)}$. A close inspection of the proofs in Sections 3 and 4 will reveal that they make no use of the dimension of the point set V. Thus, Theorems 1–3 hold also in d dimensions, with k_θ replaced by $k_{d,\theta}$.

Theorem 3. *Let V be any set of n points in $\mathbb{R}^d$. Then V has an ordered θ-graph in which each vertex has degree at most $k_{d,\theta}(H_{n-1}+1)$. Furthermore, this ordered θ-graph can be constructed in $O(k_{d,\theta} n \log^{d-1} n)$ time.*

Theorem 4. *Let V be a set of n points in $\mathbb{R}^d$ and let $G = (V, E)$ be an ordered θ-graph of V obtained by taking the points of V in random order. Then the probability that there exists a pair $u, v \in V$ such that the θ-path from u to v has more than $c \log n$ edges is $n^{-\Omega(c)}$.*

5.2 Fault-Tolerant Ordered θ-Graphs

We say that a graph $G = (V, E)$ is an f *fault-tolerant t-spanner* if $G \setminus V'$ is a t-spanner for any subset $V' \subseteq V$ of size at most f. Lukovszki [8] shows that, if we modify the construction of θ graphs so that each vertex connects to $f+1$ vertices in each cone, then we obtain an f fault-tolerant t_θ-spanner with at most $f k_\theta n$ edges.

Applying the same modification to ordered θ-graphs, i.e., connecting the vertex v to the nearest $f+1$ elements of P_v in each cone, yields the same result for ordered θ-graphs. We call such graphs f-fault tolerant ordered θ-graphs. The results of Sections 3 and 4 generalize immediately:

Theorem 5. *Let V be any set of n points in $\mathbb{R}^d$. Then V has an f-fault tolerant ordered θ-graph in which each vertex has degree at most $f k_{d,\theta}(H_{n-1}+1)$. Furthermore, this ordered θ-graph can be constructed in $O(f k_{d,\theta} n \log^{d-1} n)$ time.*

Theorem 6. *Let V be a set of n points in $\mathbb{R}^d$ and let $G = (V, E)$ be an ordered θ-graph of V obtained by taking the points of V in random order. For any $V' \subset V$ of at most f vertices in V, the probability that there exists a pair $u, v \in V \setminus V'$ such that the θ-path from u to v in $G \setminus V'$ has more than $c \log n$ edges is $n^{-\Omega(c)}$. It follows that the probability that there exists any subset $V' \subseteq V$ such that the θ-path from u to v in $G \setminus V'$ has more than $c \log n$ edges is $n^{f-\Omega(c)}$.*

6 Summary and Conclusions

We have defined ordered θ-graphs, a variant of θ-graphs that allow some flexibility in their construction. This flexibility allows us to construct spanners with low degree and spanners with low spanner diameter, but is close enough to the original definition θ-graphs that existing generalizations of θ-graphs also hold for ordered θ graphs.

We construct ordered θ-graphs by projecting points onto the walls of cones. A better spanning ratio of $(1/(1-2\sin(\theta/2)) < t_\theta$ can be obtained if, instead, we project points onto the central axes of cones. While it is possible to do this, the deletion only range tree data structure of Mehlhorn and Näher does not support the types of queries we need to perform. Using standard range trees increases the runnning time of the construction algorithm to $O(k_\theta n \log^2 n)$.

Open Problem 1. *Give an $O(k_\theta n \log n)$ time algorithm for constructing the ordered θ-graph obtained by projecting points onto central axes of cones.*

Although we have shown that every point set V of size n has an ordering in which the maximum degree of a vertex in the ordered θ-graph is $O(k_\theta \log n)$ we do not know if this result is tight.

Open Problem 2. *Does every vertex set V have an ordering π such that the π-ordered θ-graph of V has degree bounded by some function of k_θ?*

There are constructions of spanner that simultaneously have small spanner diameter and small degree [1]. Is it possible to obtain similar results using only ordered θ-graphs?

Open Problem 3. *Does every vertex set V have an ordering π such that the π-ordered θ-graph has small degree and $O(\log n)$ spanner diameter?*

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References

[1] S. Arya, G. Das, D. M. Mount, J. S. Salowe, and M. Smid. Euclidean spanners: Short, thin, and lanky. In *Proceedings of the Twenty-Seventh Annual ACM Symposium on the Theory of Computing*, pages 489–498, 1995.

[2] S. Arya, D. M. Mount, and M. Smid. Dynamic algorithms for geometric spanners of small diameter: Randomized solutions. *Computational Geometry: Theory and Applications*, 13:91–107, 1999.

[3] J. L. Bentley. Multidimensional divide-and-conquer. *Communications of the ACM*, 23(5):214–228, 1978.

[4] P. Dietz and D. Sleator. Two algorithms for maintaining order in a list. In *Proceedings of the nineteenth annual ACM Symposium on Theory of Computing, New York City, May 25–27, 1987*, pages 365–372, 1987.

[5] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley, 2nd edition, 1994.

[6] J. M. Keil. Approximating the complete Euclidean graph. In *SWAT 88: 1st Scandinavian Workshop on Algorithm Theory*, pages 208–213, 1988.

[7] J. M. Keil and C. A. Gutwin. Classes of graphs which approximate the complete Euclidean graph. *Discrete and Computational Geometry*, 7:13–28, 1992.

[8] T. Lukovszki. New results on fault-tolerant geometric spanner. In *Proceedings of the 6th International Workshop on Algorithms and Data Structures (WADS'99)*, pages 193–204, 1999.

[9] K. Mehlhorn and S. Näher. Dynamic fractional cascading. *Algorithmica*, 5:215–241, 1990.

[10] J. Ruppert and R. Seidel. Approximating the d-dimensional complete Euclidean graph. In *Proceedings of the Third Canadian Conference on Computational Geometry (CCCG'91)*, pages 207–210, 1991.