Aperture Angle Optimization Problems

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Abstract

Let P and Q be two disjoint convex polygons in the plane with m and n vertices, respectively. Given a point x in P, the aperture angle of x with respect to Q is defined as the angle subtended by the cone that contains Q, has apex at x, and has its two rays emanating from x tangent to Q. We present algorithms with complexities $O(n \log m)$ and $O(n+m)$ for computing the maximum aperture angle with respect to Q when x is allowed to vary in P. To compute the minimum aperture angle we modify the latter algorithm obtaining an $O(n+m)$ algorithm. In fact, this is optimal as we show that $\Omega(\max\{m,n\})$ is a lower bound for the minimization problem. Finally, we establish an $\Omega(n)$ time lower bound for the maximization problem.

1 Introduction

Visibility plays a singular role in computer graphics, robotics, computer vision, operations research and several other disciplines of computing science and computer engineering [8], [11]. The traditional model of visibility investigated in computational geometry allows for a guard or camera to *see in all directions*, i.e., the aperture angle is idealized to be 360 degrees. More recently, computational geometry research has begun investigating more realistic models of visibility where the aperture angle (or field-of-view angle as it is called in robotics [4], [5]) is restricted to be some angle θ less than 360 degrees. For example, given a convex polygon and a camera with aperture angle θ situated outside the polygon, Teichman [12] computes a description of all the points in space where a camera may be placed in such a way that the polygon lies completely in the field of vision of a camera with aperture angle θ. A member x of a set of points S is said to be θ-visible if a camera with aperture angle θ can be placed on x in such a way that no other member of S lies in the camera's field of vision. Avis et al. [1] obtained optimal algorithms for finding all the θ-visible points in a set S. Devroye and Toussaint [6] investigate the cardinality of the θ-visible points among a set of special points which are the intersections of a set of random lines. Finally, in another variant of the problem, Bose et al. [2] have shown that n cameras, each with specified aperture angle not exceeding 180 degrees, can be placed at n fixed locations in the plane to see the entire plane if and only if the aperture angles sum to at least 360 degrees.

The simplest of these types of problems is often found as an exercise in calculus texts and called the *picture-on-the-wall* problem (see for example [10], p. 427, problem # 20). In this problem a picture hangs on the wall in a museum above the level of an observer's eye. How far from the wall should the observer stand to maximize the angle subtended at the observer's eye by the top and bottom of the picture? While this problem is easily solved with calculus, an elegant solution that does not use calculus has been known for some time [7]. This same solution holds for the more general problem where the picture may not be orthogonal to the floor [13].

In this paper we consider a generalization of the *picture-on-the-wall* problem, namely, the problem of computing the aperture angle of a camera that is allowed to travel in a convex region in the plane and is required to maintain some other convex region within its field of view at all times. More specifically, let P and Q be two disjoint convex polygons in the plane with m and n vertices, respectively. Given a point x in P, the aperture angle of x with respect to Q, denoted $\theta(x)$, is defined as the angle subtended by the cone that contains Q, has apex at x, and has its two rays emanating from x tangent to Q. Note that Q need not be a convex polygon, however, since a cone containing Q also contains its convex hull, we can restrict our attention to convex polygons. We present an $O(n+m)$ time algorithm for computing the minimum aperture angle with respect to Q when x is allowed to vary in P. This is

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optimal as we present an $\Omega(\max\{m,n\})$ lower bound for the problem of computing the minimum aperture angle. We also present algorithms with complexities $O(n+m)$ and $O(n\log m)$ for computing the maximum aperture angle with respect to Q. Thus, when $m = o(n\log n)$ the first algorithm is faster than the second one. However, if $m = \Omega(n\log^{1+\epsilon} n)$, for any $\epsilon > 0$, the second one is faster. Finally, we prove an $\Omega(n)$ time lower bound for the maximization problem. Most proofs are omitted in this extended abstract. For full detailed proofs, we refer the reader to the technical report [3].

2 Geometric Preliminaries

In this section, we develop the geometric tools used to solve the different problems mentioned in the introduction. The model of computation used throughout this paper is the extended real RAM (for details refer to [9]).

Given a convex set C, let ∂C, $int(C)$, and $ext(C)$ denote the boundary, interior and exterior of C, respectively. Let $\angle(abc)$ denote the angle at b in triangle abc. Let $[ab]$ and (ab) denote, respectively, the closed and open line segments between two points a and b.

To gain insight into a general solution of the problems at hand, we first analyze simplified versions of the main problem. Solutions to these simplified problems will form the basis upon which the general algorithms will be constructed. The first simplification will be referred to as the Segment-Polygon Problem, where the region the camera is in is still the convex polygon P, but the convex polygon Q (the object that must kept in the field of view) is replaced by a line segment $[ab]$.

2.1 The Segment-Polygon Problem

Problem Statement: Find a point x in a convex polygon P such that $\theta(x)$ is maximum with respect to a given segment $[ab]$ that does not intersect P ($\theta(x)$ is the angle at x in triangle abx).

In order to present the solution to this problem, we first define some geometric concepts related to the solution. First of all, unless stated otherwise, we always assume that the vertices of a polygon are given in counterclockwise order.

Definition 2.1 *A line L is a critical separating line of support of P and $[ab]$ if it separates P from $[ab]$, and it is tangent to both P and $[ab]$.*

See Figure 1 for the definitions to follow. Let the critical separating lines of support to P and $[ab]$ be tangent at $\{p_j, a\}$ and $\{p_i, b\}$ respectively. These lines partition the boundary of P into two chains, and also partition the plane into four regions (cones), two of which are empty, one which contains P and the other $[ab]$. Denote the region containing P by R_P. Now, the line segment $[p_ip_j]$ partitions R_P into a triangle and an unbounded region. The chain $(p_i, p_{i+1}, ..., p_j)$ contained in the triangle is referred to as the interior separating boundary of P with respect to $[ab]$, denoted by $ISB(P)$. The complementary chain $\partial P - ISB(P)$ is denoted by $ISB(P)^c$. Note that p_i and p_j will also be assumed to be contained in the complement, $ISB(P)^c$.

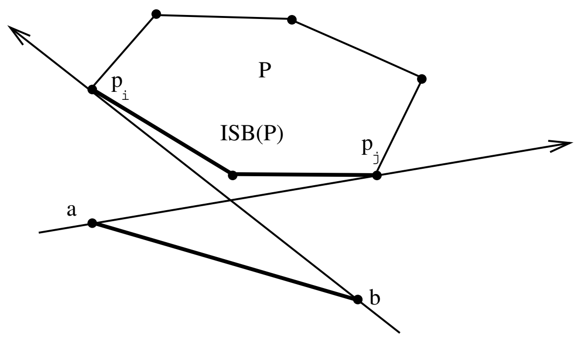

Figure 1: Illustration of definitions.

If the line $L(a,b)$ passing through $[ab]$ intersects $int(P)$, then the chain $ISB(P)$ is contained in the triangle (p_i, c, p_j), where p_i and p_j are the two tangent points as defined above and c is the extreme point of the segment $[ab]$ that is closer to P (using the definition of distance from a point b to a polygon P as $\min\{d(b,x) \mid x \in P\}$ and d is the euclidean distance). Thus, $L(a,b)$ divides the convex polygon P into two convex polygons P_1 and P_2, where $L(a,b)$ does not intersect the interior of either and $ISB(P)$ is partitioned into $ISB(P_1)$ and $ISB(P_2)$. Furthermore, the solution to our problem for P will be the maximum of the solutions obtained for the two problems on P_1 and P_2 separately since on $L(a,b)$ the maximum aperture angle is zero. Therefore, to solve the Segment-Polygon problem, we may assume that $L(a,b)$ does not intersect $int(P)$.

Lemma 2.1 *The maximum aperture angle is reached at a unique point $x \in ISB(P)$.*

Lemma 2.1 establishes the existence of a unique global maximum over $ISB(P)$. However, this does not preclude the existence of other possible local maxima. Fortunately, $\theta(x)$ is an upwards unimodal function over $ISB(P)$.

Lemma 2.2 *The function $\theta(x)$ with respect to the segment $[ab]$ is upwards unimodal over $ISB(P)$.*

The importance of showing that $\theta(x)$ is an upwards unimodal function becomes evident when developing an algorithm to obtain the maximum. Upward unimodality allows one to use binary search. The algorithm to find the maximum aperture angle follows from the discussion and is presented below. We assume in this paper that the vertices of the polygons are stored in arrays.

Algorithm 1: *Compute maximum aperture angle in a convex polygon with respect to a segment.*

Input: A convex polygon P and a segment $[ab]$ such that $L(a,b)$ does not intersect P.

Output: A point $x \in P$ such that $\theta(x)$, with respect to $[ab]$, is maximum over P.

1. Compute the chain $ISB(P)$.
2. Determine the point x, where the circle C through a and b is tangent to P, by using binary search over $ISB(P)$.
3. Exit with x.

Lemma 2.3 *Algorithm 1 finds in $O(\log m)$ time a point $x \in P$, such that $\theta(x)$ is maximum with respect to the segment $[ab]$.*

We now turn our attention to the minimization version of the Segment-Polygon problem. In order to present a complete characterization of the solution to this problem, we first define some additional geometric concepts.

Definition 2.2 *A line L is a common tangent of P and $[ab]$ if it is tangent to P and $[ab]$, and it leaves P and $[ab]$ in one of the closed halfplanes defined by L.*

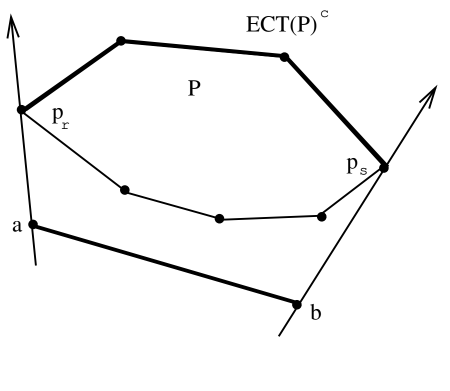

Figure 2: More definitions.

Let the common tangents of P and $[ab]$ be tangents at $\{p_r, a\}$ and $\{p_s, b\}$, respectively (see Figure 2). We define $ECT(P)$ to be the part of the boundary of P contained in the convex polygon (p_r, a, b, p_s). Let $ECT(P)^c$ be the complementary chain $\partial P - ECT(P)$. We may assume that $L(a,b)$ does not intersect $int(P)$, since we may partition $ECT(P)$ into the corresponding chains $ECT(P_1)$ and $ECT(P_2)$ as we did for the maximization problem.

Lemma 2.4 *Any point x in P where the aperture angle reaches the minimum value lies on a vertex of the chain $ECT(P)^c$.*

To find the minimum aperture angle with respect to $[ab]$, we evaluate $\theta(x)$ at the endpoints of every edge in $ECT(P)^c$ and select the global minimum. We conclude with the following.

Lemma 2.5 *In $O(n)$ time a point $z \in P$ can be found such that $\theta(z)$ is minimum with respect to the segment $[ab]$.*

2.2 The Polygon-Line Problem

We now take a second step towards the general problem and study a simplification referred to as the Polygon-Line Problem, where the object that must kept in the field of view is a convex polygon Q, but the region where the camera is allowed to lie is a line L.
Problem Statement: Given a convex polygon Q and a line L, find a point $x \in L$ such that the aperture angle $\theta(x)$ is maximum.

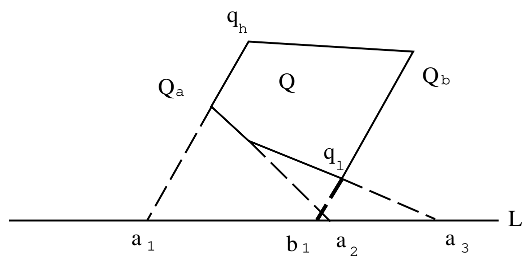

Figure 3: Partition of line L

To simplify the notation, we assume that no edge of Q is parallel to the line L. We also assume that the polygon and the line do not intersect. Without loss of generality, assume L is the X-axis, and let q_h be the vertex of Q with the highest y-coordinate and q_l be vertex with the lowest y-coordinate (see Figure 3). Thus the boundary of Q is decomposed into a left chain $Q_a = \{q_h, q_{h+1}, ..., q_l\}$ and a right chain $Q_b = \{q_l, q_{l+1}, ..., q_h\}$. We partition L by extending every edge of Q_a until it intersects L at a point a_i and every edge of Q_b until it intersects L at a point b_j. Finally we merge the ordered sets $A = \{a_1, a_2, ..., a_{l-h}\}$ and $B = \{b_1, b_2, ..., b_{n-l}\}$ (subindex addition is done modulo n) to obtain an ordered set $R = \{r_1, r_2, ..., r_n\}$.

The partition of L consists of the intervals $I_k = [r_k, r_{k+1}]$, $k = 1, 2, \ldots, n-1$, together with two unbounded intervals $I_0 = (-\infty, r_1]$ and $I_n = [r_n, +\infty)$.

The following lemma provides the link between the Segment-Line problem and the Polygon-Line problem.

Lemma 2.6 *For every interval $I_k = [r_k, r_{k+1}]$ in the partition, there are two vertices $q_s \in Q_a$ and $q_t \in Q_b$, that determine a diagonal $d_k = [q_s q_t]$ of Q, such that for every point $x \in I_k$, the aperture angle $\theta(x)$ with respect to Q is given by $\angle(q_s x q_t)$.*

As a consequence of Lemma 2.6 the aperture angle function $\theta(x)$ with respect to Q is piece-wise defined over L. For every interval I_k, the problem is reduced to the Segment-Line problem, where the segment is determined by the diagonal d_k of Q. Therefore, to find the maximum (resp. minimum), we simply compute candidates for the maximum (resp. minimum) for every interval and choose, as the global maximum (resp. minimum), the maximum (resp. minimum) of all the candidates.

Theorem 2.1 *In $O(n)$ time a point $x \in L$ and $y \in L$ can be found such that $\theta(x)$ is maximum and $\theta(y)$ is minimum with respect to Q.*

3 Two Convex Polygons

We now have the tools to solve the general problem where the object that must kept in the field of view is a convex polygon Q, and the region where the camera can lie is a convex polygon P. We assume that $P = [p_1, p_2, \ldots, p_m]$ and $Q = [q_1, q_2, \ldots, q_n]$ are each represented by an array in counter-clockwise order.

Problem Statement: Given two disjoint polygons P and Q in the plane with m and n vertices respectively, find points $x, z \in P$ such that $\theta(x)$ is maximum and $\theta(z)$ is minimum.

3.1 The Maximization Problem

Lemma 3.1 *A point $x \in P$ where the aperture angle reaches the maximum value must lie on the chain $ISB(P)$.*

Given that the maximum aperture angle is reached at a point on $ISB(P)$, we define a partition of $ISB(P)$, similar to the partition of the line in the Polygon-Line problem. Let $ISB(P) = (p_i, p_{i+1}, \ldots, p_j)$. Notice that the extension of the edges of $ISB(Q)$ and $ECT(Q)^c$ do not intersect P. Therefore, we only consider the extension of edges in $\partial Q - (ISB(Q) + ECT(Q)^c)$. This divides the boundary of Q into two chains, Q_a and Q_b. We denote by A the ordered set of intersection points between the extended edges of Q_a and $ISB(P)$, and by B the ordered set of intersection points between the extended edges of Q_b and $ISB(P)$. The ordered set R of $ISB(P)$ is determined by merging the two ordered sets A and B. The counter-clockwise order of the points of R is $(r_1 = p_i, r_2, r_3, \ldots, r_s = p_j)$. Let R_k represent the piece of the boundary of $ISB(P)$ between r_k and r_{k+1}. By construction, we see that for every point x in R_k, the aperture angle $\theta(x)$ is determined by a diagonal of Q formed by a vertex of Q in Q_a and one in Q_b.

Lemma 3.2 *For every polygonal chain $R_k \subseteq ISB(P)$ in the partition of $ISB(P)$, there are two vertices $q_r \in Q_a$ and $q_t \in Q_b$ such that for every point $x \in R_k$, the aperture angle $\theta(x)$ with respect to Q is given by $\angle(q_r x q_t)$.*

As a consequence of Lemma 3.2, the aperture angle function $\theta(x)$ with respect to Q is piece-wise defined over the chain $ISB(P)$ and for every region R_k, the problem is reduced to finding the maximum aperture angle between a convex chain and a line segment. Therefore, to find the maximum, it suffices to find the global maximum among the candidate maxima obtained for each region R_k. We summarize the algorithm below.

Algorithm 2: *Maximum aperture angle*

Input: A convex polygon Q with n vertices and a convex polygon P with m vertices. $P \cap Q$ is empty.

Output: A point x in P such that the aperture angle $\theta(x)$, with respect to Q, is maximum.

1. Find the partititon of $ISB(P)$ into chains $R_1, R_2, \ldots, R_s$.
2. For each convex chain R_k find the diagonal $[q_r q_t]$ in Q such that for every point $x \in R_k$, the aperture angle $\theta(x)$ with respect to Q is given by $\angle(q_r x q_t)$.
3. For each chain R_k, find $x_k \in R_k$ such that the aperture angle $\theta(x_k)$ is maximum over all points in R_k.
4. Exit with the maximum of all maxima computed in the previous step.

The complexity of the algorithm is dominated by the first step which is to partititon $ISB(P)$ into chains $R_1, R_2, \ldots, R_s$. We apply two different techniques to compute the partition. The first approach takes advantage of the fact that the intersections of the extensions of the edges of Q_a (resp. Q_b) with $ISB(P)$ are ordered along $ISB(P)$. Suppose that Q_a, Q_b and $ISB(P)$ have been computed. The intersections of the extensions of Q_a (resp. Q_b) with $ISB(P)$, can be computed by walking across $ISB(P)$ once. Therefore, the partition of $ISB(P)$ can be computed in $O(n + m)$ time.

The second approach takes advantage of the following. Observe that when an edge $[q_j q_{j+1}]$ in $ECT(Q)^c$

is extended, it does not intersect ∂P, and $(q_j q_{j+1} p)$ makes a *left turn* for any point p in P. When an edge (q_j, q_{j+1}) in $ISB(Q)$ is extended, it does not intersect ∂P and $(q_j q_{j+1} p)$ makes a *right turn* for any point p in P. If an edge (q_j, q_{j+1}) of Q_a is extended, then the ray from q_j in direction of q_{j+1} intersects ∂P at two points $a_k \in ECT(P)$ and $z_k \in ECT(P)^c$, that together with q_j and q_{j+1} are ordered as q_j-q_{j+1}-a_k-z_k along the ray. Finally, if an edge (q_j, q_{j+1}) in Q_b is extended, then the ray from q_{j+1} in direction of q_j intersects ∂P in two points $b_k \in ECT(P)$ and $z_k \in ECT(P)^c$, that together with q_j and q_{j+1} are ordered as z_k-b_k-q_{j+1}-q_j along the ray. These properties satisfied by the edges of Q give us another characterization of the chains determined by the support vertices of the common and separating tangents. Therefore, we do not need to know a priori the support vertices of Q, for decomposing ∂Q into $ECT(Q)^c$, Q_a, $ISB(Q)$ and Q_b. Since we can compute the intersection of a line with convex polygon P in $O(\log m)$ time, this alternate characterization provides an $O(n \log m)$ time algorithm to compute the partition of $ISB(P)$.

Let $f(n,m)$ denote the function that counts the number of operations performed on steps 2,3, and 4 of algorithm 2.

Lemma 3.3 *$f(n,m) \in O(n+m)$ and $f(n,m) \in O(n \log m)$*

Therefore, we conclude with the following.

Theorem 3.1 *Algorithm 2 finds a point $x \in P$, such that $\theta(x)$ is a maximum with respect to Q in either $O(n+m)$ or $O(n \log m)$ time.*

3.2 The Minimization Problem

We characterize the subset of points of the boundary of P where the minimum value $\theta(x)$ can be attained. In the following lemma, we establish that the points are on $ECT(P)^c$. Again, we define a partition of $ECT(P)^c$ similar to the partition of of $ISB(P)$ above. We intersect the extentions of the edges of Q_a and Q_b with $ECT(P)^c$. Let R represent the partition of the chain $ECT(P)^c$.

Lemma 3.4 *Any point x in P where the aperture angle reaches the minimum value lies on a vertex of the chain $ECT(P)^c$ or on a parition point of R.*

The lemma suggests the following algorithm.

Algorithm 3: *Minimum aperture angle*

Input: A convex polygon Q with n vertices and a convex polygon P that does not intersect Q.

Output: A point z in P such that the aperture angle $\theta(z)$, with respect to Q, is minimum.

1. Find the partititon of $ECT(P)^c$ into chains $R_1, R_2, \ldots, R_s$.
2. For each R_k find the diagonal $[q_r q_t]$ in Q such that for every point $x \in R_k$, the aperture angle $\theta(x)$ with respect to Q is given by $\angle(q_r x q_t)$.
3. For each chain R_k, find $x_k \in R_k$ such that the aperture angle $\theta(x_k)$ is minimum over all points in R_k. This is done by verifying all vertices of P on R_k as well as r_k and r_{k+1}.
4. Exit with the global minimum of all minima computed in the previous step.

Since $O(n+m)$ points are verified in order to find the minimum, we conclude with the following.

Theorem 3.2 *Algorithm 3 finds a point z in P, such that $\theta(z)$ is a minimum with respect to Q in $O(n+m)$ time.*

4 Lower Bounds

In this section, we show that the complexity of computing the minimum aperture angle of an m vertex convex polygon P with respect to an n vertex convex polygon Q has lower bound $\Omega(\max\{m,n\})$ and computing the maximum aperture angle has lower bound $\Omega(n)$. Our lower bounds rely on the fact that the polygons are given in the form of linear arrays, a very natural representation.

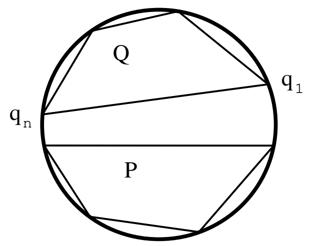

Figure 4: Lower bound construction.

For the first construction we create two convex polygons the vertices of which lie on the unit circle C centered at the origin. For P we choose its vertices on the lower semi-circle of C, whereas Q's vertices are located in the upper semi-circle (see Figure 4). These polygons have the property that every edge of each of the two polygons can be extended by an arbitrarily large distance without intersecting the interior of any other edge in either polygon. Therefore polygon Q has the appearance of a line segment to a viewer in P. In particular, Q behaves as if it were the edge $[q_n, q_1]$. Therefore, by Lemma 2.4, the minimum aperture angle must be realized by a vertex of P. Furthermore, note that since P's vertices are on the circle C and edge $[q_n, q_1]$ is a chord of the same circle, it follows that the aperture angle at each vertex of P is equal. To finish the construction we take an arbitrary vertex v of P and pull it outside the circle C but by an amount small enough to preserve the convexity of P. This arbitrary vertex will yield the smallest value aperture angle. It follows that any algorithm that omits even a single vertex in its search for the global minimum could fail on this polygon. We have therefore proved the following theorem.

Theorem 4.1 *The complexity of computing minimum aperture angle is* $\Omega(m)$.

We now show the construction that establishes the $\Omega(n)$ lower bound for computing the maximum. In order to do this, we construct a convex polygon Q such that the function $\theta(x)$ has n identical maxima in P. In fact, in our construction, for simplicity P will just be a line L which can be converted into a convex polygon after completion of the construction of Q.

We denote by $q_i = (a_i, b_i)$, $(0 \le i \le n-1)$ the vertices of Q. The left chain Q_a contains only two vertices q_0 and q_{n-1} and the right chain Q_b contains the vertices $q_1, q_2, \ldots, q_{n-1}$.

We define a family of lines H with the property that if a line N belongs to the family H, the intersection point of N, and the line perpendicular to N through $(0,1)$, is on the X-axis. This family is defined by $H = \{(x,y) \mid y = \alpha x - \alpha^2;\ \alpha \in E^2\}$ and its envelope is a parabola given by $E_H = \{(x,y) \mid y = x^2/4\}$.

We now present the construction. Assume that the line L coincides with the X-axis. Define the vertex $q_0 = (0,1)$. Let $T_0 = (1,0)$. To obtain the vertex q_1 consider the line $L(T_0) \in H$ that passes through T_0 and that is perpendicular to the line $L(q_0, T_0)$. Then $L(T_0)$ is tangent to E_H at the point $(2,1)$. Define $q_1 = (2,1)$. Let $T_1 = (2,0)$ be the orthogonal projection, on the X-axis, of q_1. In general, if $T_{i-1} = (2i-1, 0)$ is a point on the X-axis, there exists a line $L(T_{i-1}) \in H$ that is perpendicular to the line $L(q_0, T_{i-1})$ and tangent to E_H at $q_i = (2^i, 2^{2i-2})$. The set of vertices that form Q are defined as $[q_0, q_1, \ldots, q_{n-1}]$, where $q_0 = (0,1)$ and $q_i = (2^i, 2^{2i-2})$ for $i = 1, \ldots, n-1$. Since $[q_0, q_1, \ldots, q_{n-1}]$ lie on a parabola, Q is a convex polygon. Each $T_i = (2^i, 0)$ is a maximum over L. Therefore, Q has $\Omega(n)$ identical local maxima over the line L. Any one of these can be slightly perturbed to be the global maximum.

Theorem 4.2 *The complexity of computing the maximum aperture angle is* $\Omega(n)$.

A modification of the above construction yields a situation where there are $\Omega(n)$ identical local minima over the line L. Therefore, we conclude with the following.

Theorem 4.3 *The complexity of computing the minimum aperture angle is* $\Omega(\max\{m,n\})$.

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