Efficient Algorithms for Guarding or Illuminating the Surface of a Polyhedral Terrain

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Abstract

We present efficient polynomial time algorithms that place $\lfloor n/2 \rfloor$ vertex guards which cover the surface of an n-vertex polyhedral terrain, and similarly, $\lfloor n/3 \rfloor$ edge guards which cover the surface of an n-vertex polyhedral terrain. The time complexity of both algorithms, dominated by the cost of finding a maximum matching in a graph, is $O(n^{3/2})$.

1 Introduction

Victor Klee originally posed the problem of determining the minimum number of guards sufficient to cover the interior of an n-sided art gallery (polygon) in 1973. Chvátal showed that $\lfloor n/3 \rfloor$ guards are sufficient and sometimes necessary to cover the interior of an n-sided art gallery using a lengthy combinatorial argument [4]. Subsequently Fisk [8] gave a concise and elegant proof using the fact that the vertices of a triangulated polygon may be three-colored. Since then, there has been an explosion in this area where many different variations of this original problem have been studied. The reader is referred to the book by O'Rourke [12] and the recent survey by Shermer [15] for an overview of the area.

Most of the research has been carried out in two dimensions. Very little is known about guarding or illuminating objects in three dimensions. A step in this direction is the study of polyhedral terrains. The problem of guarding a polyhedral terrain was first investigated by deFloriani, et al. [6]. They showed that finding the minimum number of guards could be done using a set covering algorithm. Cole and Sharir [5] showed that the problem was NP-complete. Goodchild and Lee [9] and Lee [10] presented some heuristics for placing vertex guards on a terrain.

Bose et al. [2] showed that $\lfloor n/2 \rfloor$ vertex guards are always sufficient and sometimes necessary to guard an n-vertex terrain. With respect to edge guards, they establish that $\lfloor (4n-4)/13 \rfloor$ edge guards are sometimes necessary to guard the surface of an n-vertex terrain. The sufficiency result of $\lfloor n/3 \rfloor$ edge guards was proved by Everett and Rivera-Campo [7]. Both upper bounds are based on the Four color theorem, and thus, no practical polynomial time algorithm achieving these bounds seemed to exist. In [2], linear time algorithms (based on the five color theorem) are presented for placing $\lfloor 3n/5 \rfloor$ vertex guards which cover the surface of a polyhedral terrain, and $\lfloor 2n/5 \rfloor$ edge guards which cover the surface of a polyhedral terrain.

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In this paper, we close this gap by finding efficient polynomial time algorithms ($O(n^{3/2})$ time) that place $\lfloor n/2 \rfloor$ vertex guards which cover the surface of a polyhedral terrain, and similarly, $\lfloor n/3 \rfloor$ edge guards which cover the surface of a polyhedral terrain. The key behind these algorithms is an alternate proof to both upper bounds that avoids the use of the Four color theorem by relying on matchings. The time complexity of both algorithms is dominated by the time needed to find a maximum matching in a graph. To date, the most efficient algorithm to find a matching in a general graph takes $O(\sqrt{|V|}|E|)$ (Micali and Vazirani [11]). This turns out to be $O(n^{3/2})$ in our case since the graphs we deal with have $O(n)$ vertices and edges.

2 Preliminaries

We begin by reviewing some of the terminology used throughout this paper. Most of the geometric and graph theoretic terminology used is standard, and for additional details, we refer the reader to [13] and [3].

We define a terrain T as a triangulated polyhedral surface with n vertices $V = \{v_1, v_2, \ldots, v_n\}$. Each vertex v_i is specified by three real numbers (x_i, y_i, z_i) which are its cartesian coordinates and z_i is referred to as the height of vertex v_i. Let $P = \{p_1, p_2, \ldots, p_n\}$ denote the orthogonal projections of the points $V = \{v_1, v_2, \ldots, v_n\}$ on the X-Y plane, i.e., each point p_i is specified by the two real numbers (x_i, y_i). It is assumed that the set $P = \{p_1, p_2, \ldots, p_n\}$ is in general position, i.e., no three points are collinear so that the projections of the edges of the polyhedral surface onto the X-Y plane determine a triangulation of P (hence the term triangulated polyhedral surface). We refer to the triangulation as the underlying triangulated planar graph associated with the terrain. Therefore we can view a terrain T as the graph of a polyhedral function $z = F(x, y)$, defined over $CH(P)$.

Two points a, b on or above T are said to be *visible* if the line segment $\overline{ab}$ does not intersect any point strictly below T. Given a point (guard) p on or above T, the subset of points of T that are visible from p is called the visible region of T from p and is denoted by $VR(T|p)$.

Throughout this paper, we only consider problems concerning vertex and edge guards. A *vertex guard* is a guard that is only allowed to be placed at the vertices of T. An *edge guard* is a guard that is only allowed to be placed on the edges of T. A point x on T is said to be visible to an edge if there exists a point y on the edge such that x and y are visible.

A set of guards covers or illuminates the surface of a terrain if every point on the terrain is visible from at least one guard in the set. A set of guards with this property will be called a *guard set.* A *vertex guard set* is a set of vertex guards that covers a polyhedral terrain. Similarly, an *edge guard set* is a set of edge guards that covers a polyhedral terrain.

The combinatorial counterparts of these terrain guarding problems can be expressed as guarding problems on the planar triangulated graph underlying the terrain. A vertex guard on the graph can only guard the faces adjacent to that vertex, and an edge guard on the graph can only guard the faces adjacent to the endpoints of the edge. It seems difficult to show that the problem of guarding a polyhedral terrain is equivalent to the combinatorial problem of guarding the underlying planar triangulated graph. However, a valid placement of vertex (resp. edge) guards on the underlying triangulated graph is also a valid placement of vertex (resp. edge) guards on the polyhedral terrain since a guard on the terrain can always see the faces adjacent to it. Therefore, an upper bound on the number of guards used to guard a triangulated planar graph is also an upper bound for polyhedral terrains.

Traditionally in graph theory, a *k-vertex coloring* of a graph G is defined to be an assignment of one of k colors to each vertex of G such that no two distinct adjacent vertices have the same color. We break from tradition and call such a coloring a *K_2-free k-coloring* since no edge (clique of size 2) is monochromatic. A *K_d-free k-coloring* of a graph G is an assignment of one of k colors to each vertex of G such that no clique of size d in G is monochromatic. In this paper, we will concentrate on K_3-free 2-colorings of planar triangulated graphs.

Given a plane graph G, the dual of G, denoted G^*, is defined as follows: corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G is an edge e^* of G^*; two vertices f^* and g^* of G^* are joined by an edge e^* if and only if the faces f and g of G are separated by (i.e. share) the edge e in G.

3 K_3-free 2-coloring

K_3-free 2-colorings form the basis of the placement algorithms to follow. We first investigate these colorings on planar graphs and then address the issue of guard placement.

Definition 1 *A maximal planar graph is a connected planar graph in which every face is a triangle.*

Notice that if every maximal planar graph is K_3-free 2-colorable, then so is every planar graph. Therefore, in the remainder of this section we assume that our graphs are maximal planar graphs.

The Four color theorem [1] states that every planar graph is 4-vertex colorable. The Four color theorem implies that every triangulated planar graph admits a K_3-free 2-coloring. Unfortunately, this implication is not computationally satisfying since no simple algorithm to 4-vertex color a planar graph exists. Recall that the proof of the Four color theorem required 1200 hours of computer work involving the verification of several thousand cases. We will provide an alternate proof which does not rely on the Four color theorem. We begin by investigating some properties of K_3-free 2-colorings of planar graphs.

Theorem 3.1 *A K_3-free 2-coloring of a maximal planar graph G determines a perfect matching on its dual graph G^*.*

Proof: Given a K_3-free 2-coloring of G, let R (red) and B (blue) represent the two color classes. Define a set of edges M^* in G^* as follows: if the end vertices of an edge e of G are assigned the same color, then the corresponding dual edge e^* of G^* is placed in set M^*.

We now show that the set M^* represents a perfect matching in G^*. First, we show that no vertex in G^* is adjacent to two edges in M^*. Suppose such a vertex f^* existed, this would imply that the corresponding face f in G would be monochromatic which is a contradiction.

Second, we show that every vertex in G^* is adjacent to at least one edge in M^*. Suppose vertex f^* in G^* is not adjacent to any edge in M^*. Let f be the corresponding face in G. Let a, b, c be the three vertices of face f in G. Face f is not monochromatic by assumption, however, two of the three vertices must be assigned the same color. Without loss of generality, assume a and b are assigned color red. By construction of M^*, the dual of the edge ab must be in M^*, contradicting the fact that f^* is not adjacent to any edge in M^*. ■

Lemma 3.1 *The dual G^* of a maximal planar graph G is 3-regular and bridgeless (i.e. contains no cut edge).*

Proof: Since every face in G is a triangle, every vertex in G^* has degree three.

Suppose that G^* had a cut edge e^*. Note that the faces adjacent to a vertex v in G form a cycle in G^*. Let edge e with end points a and b in G be the dual of edge e^*. The faces adjacent to a form a cycle C in G^*. However, e^* is in cycle C contradicting the fact it is a cut edge. ■

Theorem 3.2 ([14], a proof also appears in [3]) *Every bridgeless (i.e. contains no cut edge) 3-regular graph has a perfect matching.*

Theorem 3.3 *Every planar graph admits a K_3-free 2-coloring.*

Proof: Let P be a planar graph. If P is not maximal, insert edges until it is maximal. Let G be this maximal planar graph and G^* its dual. By Lemma 3.1, G^* has a perfect matching. Let M^* represent a set of edges forming this perfect matching in G^*. Let M be the edges in G that are dual to the edges in M^*. Let G' be the graph G with the edges M removed. Since G is a maximal planar graph, G' is a planar graph where every face is a quadrilateral. Moreover, since the smallest non-trivial cycle in G' is of length 4, G' contains no odd cycles. Therefore, G' is bipartite.

Let R and B represent the two sets of vertices which form the bipartition of G'. Since the vertex set of G and G' is the same, this forms a partition of the vertices of G. Assign the color red to the vertices in R and blue to the vertices in B. We have defined a coloring of the vertices of G, such that any two vertices adjacent via an edge not in M must have distinct colors. Since every face of G has two such edges, we conclude that no face of G is monochromatic. Therefore, the sets R and B form a K_3-free 2-coloring of G and P. ■

4 Algorithms for Guard Placement

As noted in the preliminaries section, a valid placement of vertex (resp. edge) guards on the underlying triangulated graph is also a valid placement of vertex (resp. edge) guards on a polyhedral terrain since a guard on the terrain can always see the faces adjacent to it. Therefore, we will concentrate on the combinatorial problem.

Note that the underlying graph G of a polyhedral terrain need not be a maximal planar graph, since the outer face is not necessarily a triangle. However, we simply add the missing edges in order to transform the underlying graph into a maximal planar graph G' since the guard placement on this augmented graph G' is also a valid placement on the original graph G. Thus in remainder of this section we assume that the underlying graph is a maximal planar graph.

4.1 Vertex Guard Placement

Observation 4.1 *Let R and B be the two color classes of a K_3-free 2-coloring of the underlying triangulated graph of a polyhedral terrain. Both R and B form a valid vertex guard set.*

Based on this observation, we see that there always exists a guard set of size at most $\lfloor n/2 \rfloor$ and to compute such a valid vertex guard set, one simply needs to compute a K_3-free 2-coloring of the underlying graph. The proof of Theorem 3.3 implies the following algorithm. Note that we assume that G is maximal planar.

Algorithm: Vertex Guard Placement
Input: Underlying n vertex graph G.
Output: Vertex guard set of size at most $\lfloor n/2 \rfloor$.

1. Compute the dual G^* of G.

2. Compute a perfect matching M^* in G^*.

3. Form graph G' by removing from G edges dual to those in M^*.

4. Compute the two sets R and B forming the bipartition of graph G'.

5. Output the smaller of R and B.

All of the steps, except step 2, can be computed in $O(n)$ time. Step 2 can be performed in $O(n^{3/2})$ time [11], since the number of vertices and edges in G^* is $O(n)$. Therefore, the complexity of algorithm is $O(n^{3/2})$, dominated by the time taken to find the perfect matching.

Theorem 4.1 *In $O(n^{3/2})$ time, one can find a vertex set of size at most $\lfloor n/2 \rfloor$ that covers or illuminates the surface of an n vertex polyhedral terrain.*

4.2 Edge Guard Placement

We present an algorithm for placing $\lfloor n/3 \rfloor$ edge guards to cover the surface of an n vertex polyhedral terrain. Let G represent the underlying graph.

Our edge guard algorithm proceeds as follows. The first step in the algorithm is to compute a K_3-free 2-coloring of the vertices of G. Let the two color classes be: R and B.

Let Matching(x) denote a *maximal* matching (which is not necessarily a *maximum* matching) on the graph induced by the vertices in the color class x. Although Matching(x) does not provide a set of edges that guards the whole terrain, if we take all the edges in Matching(x) as well as one edge from each of the remaining unmatched vertices of color x then we guard the whole terrain by Observation 4.1. Let Guard(x) represent a set of edge guards obtained in this way. Also, let $|S|$ represent the number of elements in a set $|S|$.

We have the following relation: $|\text{Guard}(x)| = |x| - |\text{Matching}(x)|$. This relation holds because for each edge of the matching, we reduce the number of unmatched vertices by 2 which results in a reduction of the size of Guard by 1. We have two color classes R and B. Note that $|R| + |B| = n$. Let $GS = \text{Matching}(R) \cup \text{Matching}(B)$.

Lemma 4.2 *The set of edges in GS covers G.*

Proof: Let f be a face of G which is not covered by an edge in GS. This implies that none of the three vertices of f are adjacent to any edge in Matching(R) or Matching(B). Without loss of generality, let f have two vertices colored red. This contradicts the fact that Matching(R) is maximal. ■

Notice that the above lemma implies that we have three valid edge guard sets for graph G: Guard(R), Guard(B), and GS. In the next theorem we show that one of these three sets has size at most $\lfloor n/3 \rfloor$.

Theorem 4.2 *Given a polyhedral terrain on n vertices, there exists a set of size at most $\lfloor n/3 \rfloor$ edges that guards the terrain.*

Proof: If the size of GS is at most $\lfloor n/3 \rfloor$, then we're done. Suppose that the size of GS is greater than $\lfloor n/3 \rfloor$. Then we have the following.

$$\begin{aligned}
|\text{Guard}(R)| + |\text{Guard}(B)| &= |R| + |B| - (|\text{Matching}(R)| + |\text{Matching}(B)|) \\
&\leq n - |GS| \\
&\leq n - \lfloor n/3 \rfloor \text{ by assumption} \\
&\leq \lceil 2n/3 \rceil
\end{aligned}$$

Since the size of Guard(R) + Guard(B) is at most $\lceil 2n/3 \rceil$, one of the two must have size at most $\lfloor n/3 \rfloor$. ■

Once again, the time complexity is dominated by the time taken to find a maximum matching. Therefore, we conclude with the following theorem.

Theorem 4.3 *Given a polyhedral terrain on n vertices, $O(n^{3/2})$ time is sufficient to find a set S of edges to guard the terrain, where $\|S\| \leq \lfloor n/3 \rfloor$.*

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