Weighted ham-sandwich cuts

Prosenjit Bose*
School of Computer Science
Carleton University
jit@scs.carleton.ca

Stefan Langerman†
Département d'Informatique
Université Libre de Bruxelles
stefan.langerman@ulb.ac.be

Abstract

Let R and B be two sets of n points. A ham-sandwich cut is a line that simultaneously bisects R and B, and is known to always exist. This notion can be generalized to the case where each point $p \in R \cup B$ is associated with a weight w_p. A ham-sandwich cut can still be proved to exist, even if weights are allowed to be negative. In this paper, we present a $O(n \log n)$ algorithm to find a weighted ham-sandwich cut, but we show that deciding whether that cut is unique is 3-SUM hard.

1 Introduction

Let $R, B \subseteq \mathbb{R}^2$ be two finite point sets, with $|R| + |B| = n$. We call the elements of R the *red points* and the elements of B the *blue points*. A line L is said to *bisect* a set $S \subseteq \mathbb{R}^2$ if each of the two open halfplanes L^+ and L^- bounded by L contain no more than half of the points, i.e., if $||S \cap L^+| - |S \cap L^-|| \leq |S \cap L|$. The *ham-sandwich cut theorem* for point sets states that there always exists a line L that simultaneously bisect R and B. Such a line is called a *ham-sandwich cut*.

Megiddo [18] showed that, if the sets R and B are linearly separable (there exists a line that separates R from B) then a ham-sandwich cut can be found in $O(n)$ time. Edelsbrunner and Waupotitisch [7] modified Meggido's method and obtained an $O(n \log n)$ time algorithm for the general case. An optimal $O(n)$ time algorithm was found by Lo and Steiger [16]. In [4], it was shown that although a ham-sandwich cut can be computed in linear time, determining if that ham-sandwich cut is unique has a lower bound of $\Omega(n \log n)$. The best known algorithm for that task walks the median level of an arrangement of lines and so runs in time $O(n^{4/3} \log n)$.

The problem of computing ham-sandwich cuts in d dimensions, $d \geq 3$ has been considered by Lo et al [15]. Several generalizations of planar ham-sandwich cuts have also been proposed [1, 2, 5, 12, 13, 20, 3].

*Research supported in part by NSERC.
†Chercheur qualifié du FNRS

In this paper, we consider a generalization where every point p is associated with a weight $w_p \in \mathbb{R}$, positive or negative. Let $W(S) = \sum_{p \in S} w_p$. A line L bisects a set S of weighted points if

$$|W(S \cap L^+) - W(S \cap L^-)| \leq |W(S \cap L)|$$

and a ham-sandwich cut for R and B is a line that bisects both weighted sets simultaneously. The ham-sandwich cut theorem can be shown to hold in the weighted case, e.g. by application of the Borsuk-Ulam theorem [17].

In the next section, we give a $O(n \log n)$ algorithm that finds such a cut, but we show in section 3 that deciding if the ham-sandwich cut is unique is 3-SUM hard, and so, that it is unlikely that a $o(n^2)$ algorithm exists for this problem. Determining uniqueness, or finding all ham-sandwich cuts, can be done in $O(n^2)$ time using topological sweep algorithm in the dual arrangement.

2 Algorithm

To simplify the algorithm, we assume that the points are in general position (no three points collinear, no two points on the same vertical). This assumption can be removed using known techniques [6]. We also add a formal infinitesimal to the weight of one of the red points and one of the blue points. This ensures that the ham-sandwich cut L is incident to exactly one red point and one blue point, and that L is also a valid ham-sandwich cut for the original weights.

Duality transform. We use the standard duality transform that maps a the point $p = (a, b)$ to the line $T_p = \{(x, y) | y = ax + b\}$, and the line $L = \{(x, y) | y = mx + c\}$ to the point $(m, -c)$. This duality preserves incidences and above/below relations. The sets of points R and B then become sets of lines T_R and T_B. The bisectors of R (respectively B) correspond to a set of piecewise linear curves composed of edges of the arrangement T_R (respectively T_B). The curves are infinite or closed. A point at the intersection between a blue curve and a red curve is the dual of a ham-sandwich cut. It is easy to show, e.g. using the Borsuk-Ulam theorem, that the number of intersections between red and blue curves is odd.

Pruning Mechanism. We use the line partition pruning mechanism from [14]: given lines $L = \{\ell_1, \ldots, \ell_n\}$ in general position in R^2 and two other non-parallel lines ℓ_A and ℓ_B, not in L, nor parallel to any line in L, we write $C = \ell_A \cap \ell_B$. Every line in L crosses both ℓ_A and ℓ_B. The lines in L are partitionned into 4 sets, one for each quadrant ($I = +,+$, $II = +,-$, $III = -,-$, $IV = -,+$), depending on whether the line crosses ℓ_A above or below C, and whether it crosses ℓ_B above or below C (see Fig. 1). A line that belongs to one of these sets (quadrants) avoids the opposite quadrant. The lines ℓ_A and ℓ_B form an *α-partition* of L if each of the four groups contains at least αn lines. We use the following lemma from [14]:

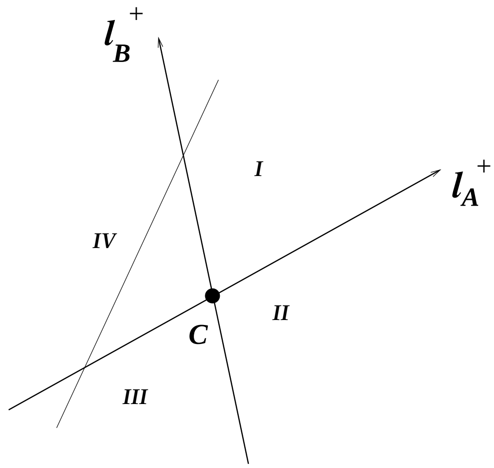

Figure 1: Pruning Mechanism

Lemma 1 *Let $L = \{\ell_1, \ldots, \ell_n\}$ be a set of n lines in general position in R^2. There exists an $(1/4)$-partition of L and it can be found in time $O(n)$.*

The algorithm will start by constructing a $(1/4)$-partition of the set of lines $T_R \cup T_B$. Each of the 4 quadrants constructed will thus be intersected by at most $3n/4$ lines.

Because the total number of intersections between the red and blue curves is odd, at least one of the four quadrants contains an odd number of those intersections. After finding that quadrant, we prune the lines it does not intersect, and recurse restricting the search to that quadrant. As we prune lines, we keep track of the restricted region in which we concentrate the search, and maintain the total weight of the pruned red and blue lines above and below that region. After each pruning step, a quarter of the lines is eliminated.

Parity. We still have to explain how to compute the parity of the intersections inside a region. We use the following lemma. A d-dimensional version of this lemma was proved by Edgar Ramos [19].

Lemma 2 *Consider a jordan curve C, a set of red curves $\mathfrak{R}$ and a set of blue curves $\mathfrak{B}$, where all curves are infinite or closed. Let $r_1, \ldots, r_k$ be the intersections of the curves of $\mathfrak{R}$ and C, in clockwise order along C, and let x_i be the number of intersections of C with the curves from $\mathfrak{B}$ between r_i and r_{i+1}. Then the parity of the number of intersections between red and blue curves inside C is the parity of $x_2 + x_4 + x_6 + \ldots + x_k$.*

This implies that we can check in $O(n \log n)$ time if the number of points dual to ham-sandwich cuts inside a given triangle is odd. This completes the algorithm.

Theorem 1 *Given two sets $R, B \subseteq \mathbb{R}^2$ of weighted points, with $|R| + |B| = n$, a weighted ham-sandwich cut can be computed in $O(n \log n)$ time.*

3 Hardness

In this section, we show that determining whether or not a weighted ham-sandwich cut is unique is a 3SUM hard problem. The class of 3SUM hard problems was initially studied by Gajentaan and Overmars [11]. Many different problems are known to be 3SUM hard. Quadratic lower bounds for 3SUM hard problems have been shown in restricted models of computation [10, 9, 8]. Thus, showing that a problem is 3SUM hard provides some evidence that there may not exist a $o(n^2)$ time solution for the problem.

For the reduction, we use the following 3SUM hard problem [11]: Given three sets A, B, and C of n integers total, determine if there are three integers $a \in A, b \in B$ and $c \in C$ such that $a + b = 2c$.

Theorem 2 *Given two sets $R, B \subseteq \mathbb{R}^2$ of weighted points, with $|R| + |B| = n$, determining whether the weighted ham-sandwich cut is unique is 3SUM hard.*

Proof: Given the three sets A, B, and C of n integers total, we produce a set of $2n+4$ weighted blue and red points, placed on three horizontal lines, one line for each set. Points in A, B, and C are placed on horizontal lines $z = 3$, $z = 1$, and $z = 2$, respectively. The points on lines A and C are colored blue, while all points on B are colored red. Each integer s is represented by a pair of points on the line corresponding to its set. One point of weight $+1$ at x coordinate $s - \varepsilon$ and one point of weight -1 at x coordinate $s + \varepsilon$, for some small $0 < \varepsilon < 1/2$. We will refer to this pair of points as a *weighted pair*. Let S be the set of $2n$ weighted points created so far.

To S, we now add four points of weight 1 far on the right of the three lines, one red point on line A, and three blue points on line B (by "far on the right", we mean that the distance from the newly added points to any point in S is at least twice the diameter of S). Thus, the total weight of the blue set of points is 3, and the total weight of the red set of points is 1. Because the total weights of both sets is odd, any ham-sandwich cut must be incident to one blue point and one red point. See Fig. 2.

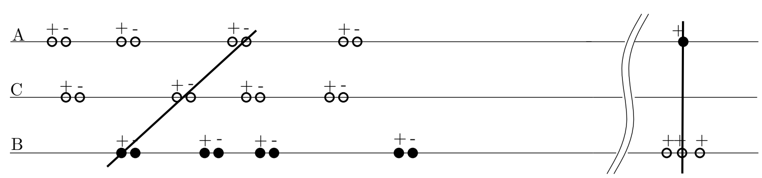

Figure 2: Uniqueness of the weighted ham-sandwich cut is 3SUM hard

We claim that the rightmost vertical line is the unique ham-sandwich cut if and only if there are no three integers $a \in A$, $b \in B$, $c \in C$, such that $a+b=2c$. Clearly, if three such integers exist, then there is a line ℓ incident to the three right points of the pair corresponding to these three elements, each having a weight of -1. The weight of the blue points is 2 to the left of ℓ, 3 to the right of ℓ and -2 on ℓ, while the weight of the red points is 1 to the left and to the right of ℓ and -1 on ℓ, so ℓ is a weighted ham-sandwich cut and so the weighted ham-sandwich cut is not unique.

If no such three integers exist, then we have to show that no line other than the rightmost vertical one is a ham-sandwich cut. To show this, consider the red and blue point incident to a candidate cut line ℓ, and first assume that the incident points are not the rightmost extra points. Now, we claim that no line ℓ can separate a weighted pair on each of the three lines. This is because the existence of such a line implies the existence of three integers integers $a \in A$, $b \in B$, $c \in C$, such that $a+b=2c$. Therefore, ℓ can separate at most two weighted pairs.

However, this implies that ℓ has blue weight 0 or 1 to the left of ℓ, $+1$ or -1 on ℓ and 2 or 3 to the right of ℓ, respectively, depending on whether the blue incident point is of weight $+1$ or -1. So ℓ cannot be a ham-sandwich cut.

If line ℓ goes through one of the blue points on the line of B, then it can only be incident to a red point by going through the red point on A (which is the claimed unique cut), or ℓ is the horizontal line corresponding to set B, which is not a ham-sandwich cut since the red weight is 0 below ℓ, 0 on ℓ, and 1 above ℓ.

If ℓ goes through the red point on A, then there are three cases: ℓ can be incident to one of the 3 points on B (which is again the claimed unique cut), or ℓ is the horizontal line corresponding to set A, (which is not a ham-sandwich cut since the blue weight is 3 below ℓ, 0 on ℓ and 0 above ℓ). In the third case, ℓ could be incident to some blue point on C. But then, the blue weight incident to ℓ is at most 1, the blue weight to the left of ℓ is 0 or 1 (if ℓ goes through a blue point of weight $+1$ or -1, respectively) and the blue weight to the right ℓ is 2 if ℓ is incident to a blue point of weight $+1$ or 3 otherwise. Since the difference is 2, ℓ is not a ham-sandwich cut. $\square$

References

[1] J. Akiyama, G. Nakamura, E. Rivera-Campo, and J. Urrutia. Perfect divisions of a cake. In *Proc. Canad. Conf. Comput. Geom. (CCCG 98)*, pages 114–115, 1998.

[2] S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink. Generalizing ham sandwich cuts to equitable subdivisions. *Discrete Comput. Geom.*, 24:605–622, 2000.

[3] P. Bose, E. D. Demaine, F. Hurtado, J. Iacono, S. Langerman, and P. Morin. Geodesic ham-sandwich cuts. In *Proc. of the 2004 ACM Symp. on Computational Geometry*, pages 1–9, 2004.

[4] H. Chien and W. Steiger. Some geometric lower bounds. In *Proc. 6th Annu. Internat. Sympos. Algorithms Comput.*, volume 1004 of *LNCS*, pages 72–81. Springer-Verlag, 1995.

[5] M. Díaz and J. O'Rourke. Ham-sandwich sectioning of polygons. In *Proc. Canad. Conf. Comput. Geom. (CCCG 90)*, pages 282–286, 1990.

[6] H. Edelsbrunner and E. P. Mücke. Simulation of Simplicity: A technique to cope with degenerate cases in geometric algorithms. *ACM Transactions on Graphics*, 9(1):66–104, 1990.

[7] H. Edelsbrunner and R. Waupotitisch. Computing a ham sandwich cut in two dimensions. *J. Symbolic Comput.*, 2:171–178, 1986.

[8] J. Erickson. Lower bounds for linear satisfiability problems. *Chicago J. Theoret. Comp. Sci.*, (8), 1999.

[9] J. Erickson. New lower bounds for convex hull problems in odd dimensions. *SIAM J. Comput.*, 28:1198–1214, 1999.

[10] J. Erickson and R. Seidel. Better lower bounds on detecting affine and spherical degeneracies. *Discrete Comput. Geom.*, 13:41–57, 1995.

[11] A. Gajentaan and M. Overmars. On a class of $o(n^2)$ problems in computational geometry. *Comput. Geom. Theory and Appl.*, 5:165–185, 1995.

[12] H. Ito, H. Uehara, and M. Yokoyama. 2-dimension ham sandwich theorem for partitioning into three convex pieces. In *Proc. Japan. Conf. Discrete Comput. Geom. (JCDCG'98)*, volume 1763 of *Lecture Notes in Computer Science*, pages 129–157. Springer-Verlag, 1998.

[13] H. Ito, H. Uehara, and M. Yokoyama. A generalization of 2-dimension ham sandwich theorem. *TIEICE: IEICE Trans. Comm./Elec./Inf./Sys.*, 2001.

[14] S. Langerman and W. Steiger. Optimization in arrangements. In *Proceedings of the 20th International Symposium on Theoretical Aspects of Computer Science (STACS 2003)*, volume 2607 of *LNCS*, pages 50–61. Springer-Verlag, 2003.

[15] C.-Y. Lo, J. Matoušek, and W. L. Steiger. Algorithms for ham-sandwich cuts. *Discrete Comput. Geom.*, 11:433–452, 1994.

[16] C.-Y. Lo and W. L. Steiger. An optimal time algorithm for ham-sandwich cuts in the plane. In *Proc. Canad. Conf. Comput. Geom. (CCCG 90)*, pages 5–9, 1990.

[17] J. Matousek. *Using the Borsuk-Ulam theorem.* Springer, 2003.

[18] N. Megiddo. Partitioning with two lines in the plane. *J. Algorithms*, pages 430–433, 1985.

[19] E. A. Ramos. Equipartition of mass distributions by hyperplanes. *Discrete Comput. Geom.*, 15(2):147–167, Feb. 1996.

[20] A. H. Stone and J. W. Tukey. Generalized 'sandwich' theorems. *Duke Math. J.*, 9:356–359, 1942.