Characterizing Proximity Trees

Prosenjit Bose* William Lenhart† Giuseppe Liotta‡

Abstract

Complete characterizations are given for those trees that can be drawn as either the relative neighborhood graph, relatively closest graph, gabriel graph or modified gabriel graph of a set of points in the plane. The characterizations give rise to linear-time algorithms for determining whether a tree has such a drawing; if such a drawing exists one can be constructed in linear time in the real RAM model. The characterization of gabriel graphs settles the conjectures of Matula and Sokal [19].

1 Introduction and Overview

Many problems in pattern recognition and classification, geographic variation analysis, geographic information systems, computational geometry, computational morphology, and computer vision use the underlying *structure* (also referred to as the *skeleton* or *internal shape*) of a set of data points revealed by means of a *proximity graph* (see for example [23], [28] [11], [19], [31] [1], [14]). A proximity graph attempts to exhibit a relation between points in a point set by connecting pairs of points that are deemed *close* by some proximity measure. It is the measure that determines the type of graph that results. Many different measures of proximity have been defined. For example, the closeness of two points can be measured by looking at the closed disk having those points as antipodal points; if the disk does not contain any other point of the point set, the two points are connected by an edge.

The resulting graph, known as the *gabriel graph*, was first introduced by Gabriel and Sokal [12]. In Figure 1 (b) the gabriel graph of the point set of Figure 1 (a) is sketched; notice that the closed disk defined by points a, b is empty, while the closed disks defined by points c, d contains point e on the boundary. Thus points c, d are not adjacent in the gabriel graph. Several papers have dealt with gabriel graphs during the last few years. Results include the paper of Matula and Sokal [19] who presented a $\Theta(n \log n)$ algorithm for computing the gabriel graph of n points in the plane. Chazelle et al. [5] showed a set of points in three-dimensional space whose gabriel graph has a quadratic number of edges. A variant of the gabriel graph, the *modified gabriel graph*, was studied by Cimikowski [4]. In a modified gabriel graph the proximity of two points in a point set is measured by determining whether the *open* disk having those points as antipodal points does not contain any other point of the set. An example is given in Figure 1 (c).

A different proximity measure is based on *lunes* instead of disks. Given two points in a point set the lune of the two points is defined as the intersection of the two disks centered at each point having radius the distance between the two points. Two points are connected by an edge when

*Research supported in part by NSERC and FCAR. School of Computer Science, McGill University, 3480 University, Montréal, Québec, H3A 2A7. jit@muff.cs.mcgill.ca

†Department of Computer Science, Williams College, Williamstown, MA 01267. This work has been done when this author was visiting the School of Computer Science of McGill University. lenhart@cs.williams.edu

‡Dipartimento di Informatica e Sistemistica, Università di Roma 'La Sapienza', via Salaria 113, I-00198 Roma, Italia. This work has been done when this author was visiting the School of Computer Science of McGill University. liotta@infokit.ing.uniroma1.it

the corresponding lune does not contain any other point of the point set. If the intersecting disks that define the lune are closed, the resulting proximity graph is known as the *relatively closest graph*, originally introduced and studied by Lankford [17]. Figure 1 (d) is the relatively closest graph of the point set of Figure 1 (a). When the lune is defined by the intersection of open disks, the result is a connected variant known as the *relative neighborhood graph* first studied by Toussaint [26]. The relative neighborhood graph of the point set of Figure 1 (a) is depicted in Figure 1 (e). Toussaint [26] studied the relationship between relative neighborhood graphs and other types of proximity graphs and showed that the relative neighborhood graph of a set of points is a supergraph of every *minimum spanning tree* (see Figure 1 (f)) and a subgraph of every *delaunay triangulation* (see Figure 1 (g)) of the set. Toussaint also gave a quadratic time algorithm to compute the relative neighborhood graph of a set of points which was subsequently improved to $\Theta(n \log n)$ by Supowit [25]. Several papers dealing with relative neighborhood graphs have since been published. Among them, Agarwal and Matoušek [1] proved a subquadratic upper bound for the number of edges of relative neighborhood graphs in the three-dimensional space. Edelsbrunner and Tan [10] exploited the properties of relative neighborhood graphs on the plane to give a quadratic time algorithm for the minmax edge length triangulation problem.

Many other proximity graphs have been studied. For example, Kirkpatrick and Radke [14] defined and studied the *β-skeleton graphs* that include both the modified gabriel and the relative neighborhood graph of a point set. Paterson and Yao [21] derived several combinatorial properties of *k-nearest neighborhood graphs* and gave bounds on the size of such graphs. Toussaint [28] introduced the *sphere-of-influence-graph* and proposed an $O(n \log n)$ algorithm for computing the sphere-of-influence graph of n points in the plane. An extensive survey on the current research in proximity graphs can be found in Jaromczyk and Toussaint [16].

In this paper we tackle the *proximity graph drawability problem*, i.e. the problem of characterizing graphs that can be drawn as certain types of proximity graphs. A graph G can be drawn as a proximity graph if there exists a set of points in the plane whose proximity graph is isomorphic to G. For example, the graph of Figure 2 (a) can be represented as a relative neighborhood graph; the corresponding representation is given in Figure 2 (b). The problems of characterizing proximity drawable graphs remains open and not many results are known to date. Recently, Lubiw and Sleumer [18] proved that maximal outerplanar graphs can be drawn both as relative neighborhood and gabriel graphs. Dillencourt and Smith [8] showed that any triangulation containing neither chords nor nonfacial triangles can be drawn as a delaunay triangulation. Di Battista and Vismara [7] characterized the allowable angles of the triangles of a delaunay triangulation through a non-linear system of inequalities. Other interesting problems deal with trees. Monma and Suri [20] showed that no tree with vertex degree greater than six can be drawn as a minimum spanning tree and provided a drawing algorithm for trees with vertex degree at most five. Recently, Eades and Whitesides [9] have proved that it is $NP-$hard to decide whether a tree with vertices of degree six can be drawn as a minimum spanning tree. Trees that can be drawn as sphere of influence graphs were characterized by Jacobson et al [15]. For a thorough overview on graph drawing problems and algorithms, the reader is referred to the excellent bibliography of Di Battista, Eades, Tamassia and Tollis [6].

We study the proximity graph drawability problem for trees. We give a complete characterization of trees that can be drawn as (1) relative neighborhood graphs, (2) relatively closest graphs, (3) gabriel graphs and (4) modified gabriel graphs. A brief overview of the results contained in the paper is listed below.

1. **Characterization of modified gabriel trees**. Cimikowski [4] proved that every tree with vertex degree at most three is drawable as a modified gabriel tree, while no tree with vertex degree greater than six can be drawn as this type of proximity graph. We complete the characterization by proving that no tree with vertices of degree greater than three can be drawn as a modified gabriel tree. The proof exploits the property that the minimum spanning tree of a

set of points is always a subgraph of the modified gabriel graph of the set.

2. **Characterization of relative neighborhood trees and relatively closest trees**. Urquhart [30] proved that no tree with vertex degree greater than six is drawable and that all trees with vertex degree at most three are drawable. We close the gap by providing a drawing algorithm for trees with vertices of degree at most five. The time complexity of the proposed algorithm is proportional to the size of the tree in a real RAM model of computation. We note that a relative neighborhood tree is also a minimum spanning tree of a given set of points. Thus, we believe that our algorithm is a simpler drawing algorithm than the one given by Monma and Suri [20].

3. **Characterization of gabriel trees**. Matula and Sokal [19] showed a drawing algorithm for trees where the vertex degree is at most three, and proved that no tree with vertex degree greater than five is drawable as a gabriel graph. They also conjectured that drawable trees cannot have vertices of degree greater than four and cannot have two adjacent vertices of degree four. Sendov [24] claimed to have proved both conjectures; unfortunately, a counter-example to Sendov's result has been recently given by Garcia [13], leaving the conjectures still open. We first give a new characterization of gabriel trees that relates the gabriel tree of a set of points to the delaunay triangulation of the set. Second, we exhibit an infinite family of trees, called *forbidden trees*, which we show cannot be drawn as gabriel graphs. Finally, we show that any non-forbidden tree can be drawn as a gabriel graph, by providing a linear time drawing algorithm in the real RAM model of computation. As a side-effect, both the conjectures of Matula and Sokal are settled in the affirmative.

4. **Drawability testing algorithms**. For each type of proximity graph studied in the paper, the proposed characterization gives rise to a linear time testing algorithm to determine whether a tree can be drawn as that type of proximity graph.

Another contribution of this paper is to exploit the interplay between different types of proximity graphs as a proof technique. In fact several geometric properties of the proximity graphs under consideration are proved by considering their combinatorial relations with other types of proximity graphs that have the property of being either supergraphs or subgraphs of the class of graphs of interest.

The paper is organized as follows: Section 2 gives necessary definitions. Section 3 completes the characterization of modified gabriel trees. Section 4 presents the characterization of relatively closest trees and relative neighborhood trees as well as describing an algorithm for drawing such trees. Section 5 does the same for gabriel trees. Closing remarks and open problems can be found in Section 6. An appendix with figures can be found at the end of the paper.

Some proofs are omitted in this extended abstract. For full detailed proofs, we refer the reader to the technical report [3].

2 Preliminaries

We begin by defining some of the graph theoretic and geometric terminology used in this paper. For more details see [2] and [22]. If T is a tree, $v \in V(T)$, and x is a neighbor of v, $T_x(v)$ denotes the connected component of $T - v$ which contains x. Similarly, $T_x[v]$ denotes $T_x(v)$ along with the vertex v and the edge xv. A *rooted tree* (T, r) is a tree T along with a distinguished vertex r (called the *root* of T).

We will be using a variety of geometric objects to describe our tree drawing algorithms. Let $\mathbb{R}^2$ denote the Euclidean plane. Given a point $p \in \mathbb{R}^2$ and a radius $\rho > 0$, $D[p, \rho]$ denotes the closed disk of radius ρ centered at p. Any sector of $D[p, \rho]$ of angle $\alpha < 2\pi$ can be specified by a vector d (with origin p) which bisects the angle of the sector; the sector is denoted by $S[p, \alpha, d, \rho]$. A sector having angle α is called an *α-sector*. A sector of infinite radius is called a *wedge* and is denoted by $W[p, \alpha, d]$. The arc which forms the common boundary of $D[p, \rho]$ and $S[p, \alpha, d, \rho]$ is denoted by $A[p, \alpha, d, \rho]$. Given a ray r with source point p, we define $HP(r, p)$ to be the closed half-plane not containing r determined by the line through p orthogonal to r. Let $d(x, y)$ denote the Euclidean distance between x and y. Finally, given three points a, b, c, in general position, $\angle abc$ denotes the *strictly smaller* angle between segment $\overline{ba}$ and segment $\overline{bc}$, unless stated otherwise.

Intuitively speaking, a *proximity graph* on a finite set of points $S \subset \mathbb{R}^2$ is obtained by connecting pairs of points of S with line segments if the points are considered to be *close* in some sense. Different definitions of closeness give rise to different proximity graphs. One technique for defining a proximity graph on a set of points is to select a geometric region defined by two points of S—for example the smallest disk containing the two points—and then specifying that a segment is drawn between the two points if and only if this region contains no other points from S. Such a region will be referred to as a *region of influence* of the two points. Five such definitions follow.

Given a set S of points in $\mathbb{R}^2$, the *relatively closest graph of S*, denoted by $RCG(S)$, has a segment between points u and v in S if the intersection of the closed disks of radius $d(u, v)$ centered at u and v is empty. This region of influence is referred to as the *closed lune* of u and v and is denoted by $L[u, v]$. Equivalently, $u, v \in S$ are adjacent if and only if

$$d(u, v) < \max[d(u, w), d(v, w)], \text{ for all } w \in S, w \neq u, v.$$

An example of such a graph is shown in Figure 1 (d). Observe that $RCG(S)$ need not be connected. The notation $RCG(S)$ is used to represent both the set of points and segments in the plane and the graph which these points and segments define. We say that a graph G is a *relatively closest graph* if G is isomorphic to $RCG(S)$ for some finite set $S \subset \mathbb{R}^2$. $RCG(S)$ is then called an RCG *drawing* of G or a *realization* of G as a relatively closest graph.

The *relative neighborhood graph of S*, denoted by $RNG(S)$, is defined similarly to $RCG(S)$, except that open disks are used instead of closed disks. That is, $RNG(S)$ is obtained by replacing $<$ with $\leq$ in the above inequality. An example of such a graph is shown in Figure 1 (e).

The *gabriel graph of S*, denoted by $GG(S)$, has as its region of influence the closed disk having segment $\overline{uv}$ as diameter. That is, two vertices $u, v \in S$ are adjacent if and only if

$$d^2(u, v) < d^2(u, w) + d^2(v, w), \text{ for all } w \in S, w \neq u, v.$$

This region is referred to as the *closed disk* of u and v, denoted as $D[u, v]$. See Figure 1 (b) for an example. A *gabriel graph* is a graph which is isomorphic to $GG(S)$ for some $S \subset \mathbb{R}^2$.

The *modified gabriel graph of S*, denoted by $MGG(S)$, is defined similarly to the gabriel graph of S, with the difference that the open disk is used instead of the closed disk; that is, $\leq$ replaces $<$ in the previous inequality. See Figure 1 (c) for an example.

Before describing the final type of proximity graph discussed in this paper, the delaunay triangulation, we present two definitions which will prove useful in our tree drawing algorithms. Let $S \in \mathbb{R}^2$ be a finite point set and let $P(S)$ be a proximity graph on S whose region of influence is determined by two points. Let X and Y be any (not necessarily finite) subsets of $\mathbb{R}^2$. We say that S *separates* X and Y if for each $x \in X$ and each $y \in Y$, there is a point in S which lies in the region of influence determined by x and y. Thus if S' contains S, $x \in X \cap S'$ and $y \in Y \cap S'$, then $P(S')$ will not contain the edge from x to y. Our algorithms will proceed by representing some of the vertices of the tree to be drawn as a set $S \in \mathbb{R}^2$ and then finding regions separated by S in which we can draw the remaining portions of the tree.

The second definition concerns a certain type of graph drawing. Given some definition of proximity, we say that a rooted tree (T, r) has an *α-drawing* if there exists a set S such that T is isomorphic

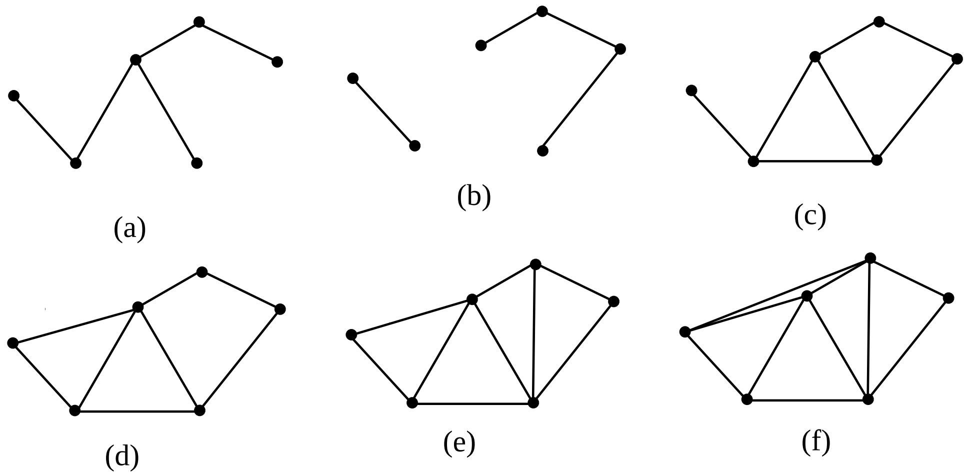

Figure 1: Six different proximity graphs on a set S of six points: (a) $MST(S)$, (b) $RCG(S)$, (c) $RNG(S)$, (d) $GG(S)$, (e) $MGG(S)$, (f) $DT(S)$.

to the proximity graph of S and S is contained in some α-sector having as apex the point of S corresponding to r. Note that the radius of the sector is irrelevant; if (T, r) has some α-drawing, it can be drawn in any α-sector of any positive radius.

A *delaunay triangulation* of a set S of points in the plane, denoted by $DT(S)$, is a triangulation of S such that for each interior face, the triangle which bounds that face has the property that the circle circumscribing the triangle contains no other points of the graph in its interior. An example of a delaunay triangulation is given in Figure 1 (g). A set S may admit more than one delaunay triangulation, but only if S contains four or more co-circular points. A list of properties of the delaunay triangulation can be found in [22].

Finally, given a set S of points in the plane, the *minimum spanning tree of* S, denoted by $MST(S)$, is the tree whose weight is no greater than the weight of any other tree on S. See Figure 1 (f) for an example.

The following relationships among the different proximity graphs hold for any finite set S of points in the plane.

Lemma 2.1 *[4]* $RCG(S) \subseteq RNG(S) \subseteq GG(S) \subseteq DT(S)$

Lemma 2.2 *[26]* $MST(S) \subseteq RNG(S)$.

Lemma 2.3 *[4]* $GG(S) \subseteq MGG(S)$

Clearly $MST(S)$ need not be a subgraph of $RCG(S)$ since the latter is not even guaranteed to be connected; if, however, $RCG(S)$ is connected, then it does contain $MST(S)$. It is worth noting that in several papers $GG(S)$ is defined using open disks instead of closed disks (i.e. in these papers $GG(S)$ is what we have called $MGG(S)$). The definition we have chosen to adopt in the present paper is the one given by Gabriel and Sokal in the paper which first defines gabriel graphs [12]. The distinction is worth noting since, for example, $GG(S)$ is guaranteed to be a subgraph of $DT(S)$ whereas $MGG(S)$ need not be (in fact, $MGG(S)$ may contain crossing segments).

3 Modified Gabriel Trees

In this section we characterize modified gabriel trees. Matula and Sokal [19] showed that for any tree T of maximum degree at most three, a GG drawing of T can be computed in time proportional to the size of T in the real RAM model. It is easy to see that the drawings they compute are also MGG drawings. Thus our task in this section is to show that a modified gabriel tree cannot have a vertex of degree greater than three. Let S be a finite set of points in the plane.

Lemma 3.1 *If $MGG(S)$ is a tree then the angle between any two adjacent edges is greater than $\pi/2$.*

Proof: We proceed by contradiction. Suppose v, x, y are vertices in S such that vx, vy are in $MGG(S)$ and $\angle xvy \leq \pi/2$. Let T_x and T_y be the subtrees, containing x and y respectively, obtained by removing v from $MGG(S)$.

Let $x' \in T_x$ and $y' \in T_y$ be two vertices such that

$$d(x', y') \leq d(\overline{x}, \overline{y}), \forall \overline{x} \in T_x \text{ and } \forall \overline{y} \in T_y.$$

D(x', y') does not contain any points of T_x or T_y by definition. It must, however, contain some point z in $MGG(S) - T_x - T_y$ since there is not an edge between x' and y' in $MGG(S)$. $MGG(S)$ is a tree, thus there must be a unique path from v to z, denoted by $P(v, z)$. This path contains no vertices of T_x or T_y, since z is in a connected subgraph of $MGG(S) - v$ different from both T_x and T_y. We can therefore conclude the following:

1. Since $MGG(S)$ is a minimum spanning tree, by Lemmas 2.1, 2.2, and 2.3, and since $P(v, z) \cup P(v, x') \cup x'z$ forms a cycle where every edge is contained in $MGG(S)$ except for $x'z$, we have that $d(x', z) > d(x, v)$.
2. $d(y', z) > d(y, v)$ by the same argument as above.
3. $d(x', y') \leq d(x, y)$ by the relation between x' and y'.
4. $\angle x'zy' > \pi/2$ since z is contained in $D(x', y')$.

The above four inequalities allow us to conclude that $\angle xvy > \pi/2$, leading to a contradiction. ■

Corollary 3.1 *A modified gabriel tree cannot have a vertex of degree greater than three.*

This completes the characterization of modified gabriel trees, which we summarize in the following theorem.

Theorem 3.1 *A tree T is a modified gabriel tree if and only if the maximum vertex degree of the tree is at most three. Furthermore, in the real RAM model, an MGG drawing of T can be computed in time proportional to the size of T.*

4 Relatively Closest and Relative Neighborhood Trees

In this section we characterize relatively closest and relative neighborhood trees. Cimikowski [4] has shown that no tree with a vertex of degree greater than five can be a relatively closest or relative neighborhood tree. We show here that any tree with maximum degree at most five is a relatively closest and a relative neighborhood tree. The following lemmas form the geometric foundations for the drawing algorithm.

Lemma 4.1 *Let $P = \{p_0, p_1, p_2\}$ be a set of points of $\mathbb{R}^2$ such that $\pi/3 < \angle p_1 p_0 p_2 < \pi$ and $d(p_0, p_1) = d(p_0, p_2)$. For any $\rho < (d(p_1, p_2) - d(p_0, p_1))/3$, $\{p\}$ separates $D[p_1, \rho]$ and $D[p_2, \rho]$ in both the relative neighborhood and relatively closest proximity graphs.*

Lemma 4.2 *Given a disk $D[p_0, r]$ and a point p on the boundary of the disk. For any $0 < \rho < r$, there exists an arc $A = A[p, \alpha, d, \rho]$, with $\alpha > \pi$ and ray d with origin p_0 bisecting α, such that for every point $x \in A$, $d(x, p_0) > d(p, p_0)$.*

Lemma 4.3 *Any tree T with maximum vertex degree five is a relatively closest (relative neighborhood) tree. Furthermore, in the real RAM model, a set S of points can be computed such that T is isomorphic to $RCG(S)$ $(RNG(S))$ in time proportional to the size of T.*

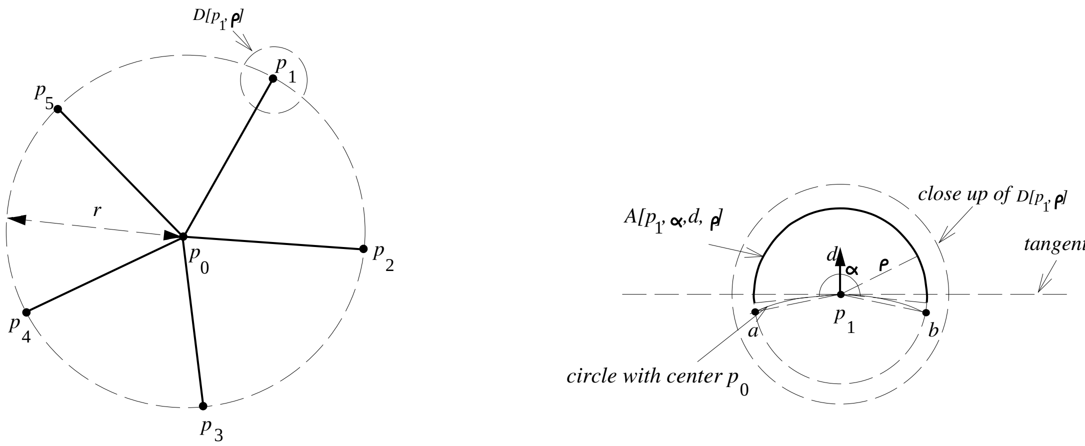

Figure 2: Illustration for Lemma 4.3.

Proof: (Sketch). Let T be a tree with maximum vertex degree five. We present a construction that draws T in the plane in such a way that, letting S denote the set of points at which the vertices of T are drawn, our drawing is identical to $RCG(S)$ and $RNG(S)$. We assume that all vertices of degree greater than one have degree five; vertices of intermediate degree can be treated in the construction as if they have degree five. Let v_0 represent a vertex of degree five; we now consider T to be a rooted tree having v_0 as root. See Figure 3 (a).

Let $v_1, \ldots, v_5$ represent the five children of vertex v_0. Then $T - v_0$ consists of five subtrees having roots $v_1, \ldots, v_5$. We first show where to place v_i with respect to the position of v_0. We then show how to place each of the subtrees rooted at v_i.

Pick any point p_0 in the plane and let p_0 represent the vertex v_0. Let $D[p_0, r]$ be a disk centered at p_0 with radius $r > 0$, and let C be the boundary of such a disk. Choose any point p_1 on C, and let p_1 represent vertex p_1. Let p_i, for $i = 2, \ldots, 5$ be the point on C such that the $\angle p_{i-1} p_0 p_i = (2\pi/5)$. Each p_i will represent the corresponding v_i. Thus, we have a set of points whose relatively closest (relative neighborhood) graph represents the vertex p_0 and its five children.

We now show how to draw the subtrees rooted at the children of vertex p_0. Let $R_i = \overline{D[p_0, r]} \cap D[p_i, \rho]$, where $D[p_i, \rho]$ is defined as in Lemma 4.1 and the bar denotes set complementation. Since the angle between any pair of children of p_0 is greater than $\pi/3$, we have that $\forall x_i \in R_i$ and $\forall x_j \in R_j$, $L(x, y)$ are separated by $\{p_0\}$, for $0 \leq i, j \leq 4, i \neq j$. Therefore no vertex v_i placed in R_i can be adjacent to any vertex v_j placed in R_j in the relatively closest (relative neighborhood) graph of the set of points being constructed. Also $\forall x \in R_i$, $0 \leq i \leq 4$, $L(p_0, x)$ contains p_i since xp_0 is the longest side of the triangle determined by x, p_o and p_i. Thus, similarly, the edge xp_0 is not in the relatively closest (relative neighborhood) graph of the set of points being constructed.

The vertex p_i can have degree at most five. Since it is already adjacent to p_0, we need only to show how to add four children to vertex p_i. Let $D[p_i, \rho']$ be a disk centred at p_i with radius $0 < \rho' < \rho$. Let $A_i = A[p_i, \alpha, d, \rho]$ be an arc defined as in Lemma 4.2 (see Figure 3 (b)). The four children of p_i are represented by four points on A_i such that the angle between any two points is greater than $\pi/3$. This is always possible since the angle α defining A_i is greater than π. Again, since we can compute a disk around each of the children of p_i that respects Lemma 4.1, we continue the construction for each child of each p_i. The total amount of work involved in computing the drawing of the graph, in the real RAM model, is proportional to the size of the graph. ■ An example

of a relative neighborhood tree produced by the algorithm implicit in the above proof is given in Figure 2. Thus we have a complete characterization of relatively closest and relative neighborhood trees, which we summarize below.

Theorem 4.1 *A tree T is a relatively closest or relative neighborhood tree if and only if the maximum vertex degree of the tree is at most five. Furthermore, in the real RAM model, an RCG or RNG drawing of T can be computed in time proportional to the size of T.*

5 Gabriel Trees

In this section, we characterize gabriel trees. Matula and Sokal [19] conjectured that a gabriel tree cannot have a vertex of degree greater than four and cannot have two adjacent vertices of degree four. Sendov [24] claimed to have proved both conjectures, however a counter-example to the result used by Sendov in his proof has been recently given by Garcia [13]. One consequence of our characterization of gabriel trees is the resolution of both of these conjectures in the affirmative.

Let S be a set of points in the plane. As in the previous section, we first establish some properties of $GG(S)$. The first property can be verified using the same argument as in the proof of Lemma 3.1.

Lemma 5.1 *If $GG(S)$ is a tree then the angle between any two adjacent edges is greater than or equal to $\pi/2$.*

Corollary 5.1 *If $GG(S)$ is a tree it cannot have a vertex of degree greater than four.*

We now investigate some properties that relate a delaunay triangulation of a set S to $GG(S)$. Let S be a finite set of points and let $DT(S)$ be a delaunay triangulation of S. For any three points x, y and z, let $D[x, y, z]$ be the smallest closed disk containing x, y and z and $D(x, y, z)$ the corresponding open disk.

Lemma 5.2 *Let ab be an edge of $DT(S)$.*

1. *If ab is contained in triangles $\triangle(abc)$ and $\triangle(abd)$ of $DT(S)$ then $ab \in GG(S)$ iff $\angle acb$ and $\angle adb$ are both less than $\pi/2$.*

2. *If ab is contained in only one triangle $\triangle(abc)$ of $DT(S)$ then $ab \in GG(S)$ iff $\angle acb < \pi/2$.*

Proof: If $\angle acb < \pi/2$ then $D[a, b] \neq D[a, b, c]$; similarly if $\angle adb < \pi/2$ then $D[a, b] \neq D[a, b, d]$. Thus $D[a, b] \subseteq D(a, b, c) \cup D(a, b, d) \cup \{a, b\}$. But since $D(a, b, c)$ and $D(a, b, d)$ contain no points of S, $D[a, b]$ contains no points of S other than a and b and therefore ab is in $GG(S)$. In the other direction we see that if either $\angle acb \geq \pi/2$ or $\angle adb \geq \pi/2$ then $D[a, b]$ contains either c or d and so $ab \notin GG(S)$. A similar argument holds in the case that ab is contained in only one triangle of $DT(S)$. ■

Theorem 5.1 *Let S be a set of points in the plane and let C represent any cycle in $DT(S)$. $GG(S)$ is a tree iff there exists an edge $ab \in C$ such that for some $p \in S$, $\triangle(apb)$ is a face of $DT(S)$, $\triangle(apb)$ lies inside C, and $\angle apb \geq \pi/2$.*

Proof: **If part**. Consider any two cycles C and C' in $DT(S)$. We will say that C is *contained* in C' if the region bounded by C' contains the region bounded by C. This gives a partial order on the cycles of $DT(S)$. Since $GG(S)$ is a tree, any cycle C must contain at least one edge that is not in $GG(S)$. Consider then the set of all cycles which do not satisfy the conclusion of the theorem. If this set is empty, then we are done; otherwise this set contains a cycle C which is maximal with respect to containment. Consider an edge ab on C which is not in $GG(S)$. Since $ab \notin GG(S)$, by Lemma 5.2 it must lie on a triangle $\triangle(apb)$ of $DT(S)$ such that $\angle apb \geq \pi/2$. Since ab does not satisfy the conclusion of the Theorem, $\triangle(apb)$ must lie outside C. Let C' be the smallest (in terms of containment) cycle of $DT(S)$ which contains both C and $\triangle(apb)$. Then the edges of C' consist of edges of C and at least one of ap and bp. Thus C' violates the conclusion of the theorem, since clearly the edges ap and pb do not satisfy the conclusion of the theorem. This, however, contradicts the maximality of C.

Only if part. For every cycle in $DT(S)$, edge ab does not belong to $GG(S)$. Since, for Lemma 2.1 $GG(S) \subseteq DT(S)$, $GG(S)$ is a tree. ■

It is the relationship between gabriel trees and delaunay triangulations described in Theorem 5.1 that allows us to extract the following key forbidden configurations.

Lemma 5.3 *If $GG(S)$ is a tree then it cannot contain as a subtree a path*

1. *$P = abcd$ such that $\angle abc = \pi/2$ and $\angle acd < \pi/2$.*
2. *$P = abcde$ such that $\angle bcd = \pi/2$, $\angle abd = \pi/2$ and $\angle bde = \pi/2$.*
3. *$P = abcde$ such that $\angle abc = \pi/2$, $\angle cde = \pi/2$ and $\angle ace < \pi/2$.*

These simple geometric constraints translate into important graph theoretic constraints that play a vital role in the characterization of gabriel trees. The following lemma and following three corollaries are all based on Lemma 5.3. In the rest all subscripts are considered modulo 4.

Lemma 5.4 *Assume that $GG(S)$ is a tree. Let v be a vertex of degree four in $GG(S)$ and let $v_i, 0 \leq i \leq 3$ be the neighbors of v in clockwise order. If some v_i has degree three then $\angle v_{i-1}v_iv_{i+1} \leq \pi/2$.*

Corollary 5.2 *A gabriel tree cannot contain two adjacent vertices of degree four.*

Observe that corollaries 5.1 and 5.2 together settle the conjectures of Matula and Sokal.

Corollary 5.3 *If $GG(S)$ is a tree it cannot contain a vertex of degree four adjacent to four vertices of degree three.*

The forbidden subtree described in Corollary 5.3 is just the smallest member of an infinite family of forbidden subtrees for gabriel trees. Some of the relationships among these ideas can be seen in the next corollary.

Corollary 5.4 *If $GG(S)$ is a tree and contains a subtree T' consisting of a vertex v of degree four adjacent to three vertices of degree three and a fourth vertex v_0 of degree one then (T', v_0) does not have an α-drawing for any $\alpha \leq \pi/2$.*

Proof: Let v_0, v_1, v_2 and v_3 be the neighbors of v in clockwise order, where v_1, v_2 and v_3 all have degree three (see Figure 4). Then, by Lemma 5.4, for $i = 1, 2, 3$, $\angle v_{i-1}v_iv_{i+1} \leq \pi/2$. In fact, to avoid one of the configurations of Lemma 5.3, at least one of these three angles must be strictly less than $\pi/2$. Thus $\angle v_3v_0v_1 > \pi/2$ and so (T', v_0) cannot be α-drawn in any sector having apex v_0 and $\alpha \leq \pi/2$. ∎

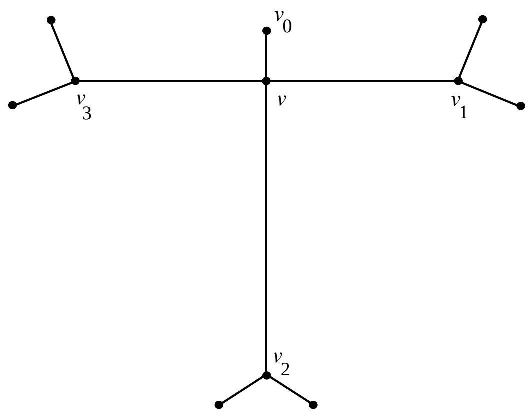

Figure 3: A wide tree.

It turns out that rooted trees which can only be drawn in α-sectors where $\alpha > \pi/2$ play an important role in characterizing those trees which are not gabriel trees. To describe this class of trees completely, we need the following definition.

Definition 5.1 A rooted tree (T, r) is *wide* if

1. $deg(r) = 2$ or
2. $deg(r) = 1$, r has a neighbor v of degree four and each rooted tree $(T_x(v), x), x \neq r$, is wide.

Lemma 5.5 *Let (T, v_0) be a wide rooted tree with $deg(v_0) = 1$. Then (T, v_0) has no α-drawing for any $\alpha \leq \pi/2$.*

Proof: We will actually show the following: Given a realization of (T, v_0) as a gabriel tree, letting v denote the neighbor of v_0 and letting v_0, v_1, v_2 and v_3 denote the neighbors of v in clockwise order then $\angle v_3v_0v_1 > \pi/2$. From this the lemma follows.

We proceed by induction on the number of vertices in (T, v_0). The base case is the eleven vertex tree with vertex v_0 adjacent to vertex v of degree four and such that each neighbor of v other than v_0 has two neighbors of degree one (see Figure 4). The truth of the lemma for this graph follows from Corollary 5.4. Suppose now the statement is true for any tree with fewer than n vertices. We prove it for trees having n vertices. If v_1, v_2, and v_3 have degree three, then again the statement is true by Corollary 5.4. Suppose a vertex adjacent to v, say v_2 has degree two. Since $(T_{v_2}(v), v_2)$ is wide, the other neighbor of v_2 must have degree 4. Call this neighbor w and let w_1 and w_2 be the neighbors of w which come before and after v_2 in clockwise order around w. From Lemma 5.3, we conclude that $\angle w_1 v_2 v_1 \geq \pi/2$ and $\angle w_2 v_2 v_3 \geq \pi/2$. By induction, $\angle w_2 v_2 w_1 > \pi/2$. Thus $\angle v_3 v_2 v_1 < \pi/2$. Also, angles $\angle v_2 v_1 v_0$ and $\angle v_0 v_3 v_2$ are each at most $\pi/2$ by Lemma 5.4. Therefore, $\angle v_1 v_0 v_3 > \pi/2$, since v_0, v_1, v_2, v_3 form the vertices of a convex quadrilateral. A similar argument holds if v_1 and/or v_3 has degree two. ■

We are now in a position to describe the class of forbidden trees.

Definition 5.2 A tree T is *forbidden* if it contains any of the following

1. A vertex of degree at least five
2. Two adjacent vertices of degree four
3. A vertex v of degree four such that for each neighbor x of v, the subtree $(T_x(v), x)$ is wide.

If, for example, two leaves are added to vertex v_0 of Figure 4, the result is a forbidden tree.

Lemma 5.6 *No forbidden tree T can be realized as a gabriel graph.*

Proof: If T has a vertex of degree greater than four, then by Corollary 5.1 it cannot be realized as a gabriel tree. Similarly, by Corollary 5.2, T cannot be realized as a gabriel tree if it has two adjacent vertices of degree four. Suppose now that T contains a vertex v of degree four with neighbors v_0, v_1, v_2, and v_3, such that the subtrees $(T_{v_i}(v), v_i), (i = 0 \ldots 3)$ rooted at v_i are wide. Suppose T has been realized as a gabriel tree. As was shown in the proof of Lemma 5.5, for each $i = 0 \ldots 3$, $\angle v_{i-1} v_i v_{i+1} > \pi/2$. But this is impossible since $v_0, ..., v_3$ form the vertices of a convex quadrilateral. ■

To complete our characterization of gabriel trees, we need to establish the converse of Lemma 5.6. To do so we will describe a drawing algorithm which is guaranteed to produce, for any non-forbidden tree, a realization as a gabriel tree. The following geometric tool forms the basis of the drawing algorithm.

Lemma 5.7 *Let $P = \{p, p_0, p_1, p_2, p_3\} \subset \mathbb{R}^2$ be such that*

1. *$\angle p_i p p_{i+1} = \pi/2$, for $i = 0, \ldots, 3$*
2. *$\angle p_3 p_0 p_1 > \pi/2$ and*
3. *$\angle p_{i-1} p_i p_{i+1} < \pi/2$, for $i = 1, \ldots, 3$*

There exist angles β_i, vectors d_i and radii $\rho_i > 0$, for $i = 0, \ldots, 3$, such that

1. *$\beta_0 > 0$*
2. *$\beta_i > \pi/2$, for $i = 1, \ldots 3$*
3. *P separates each pair $S_i = S[p_i, \beta_i, d_i, \rho_i]$ and $S_j = S[p_j, \beta_j, d_j, \rho_j]$, $i \neq j$.*

Proof: See Figure 5. Let Q denote the convex quadrilateral having vertices $p_0, \ldots, p_3$. Let l_i and r_i be two vectors emanating from p_i external to Q such that

1. the clockwise angle β_i between l_i and r_i is less than π

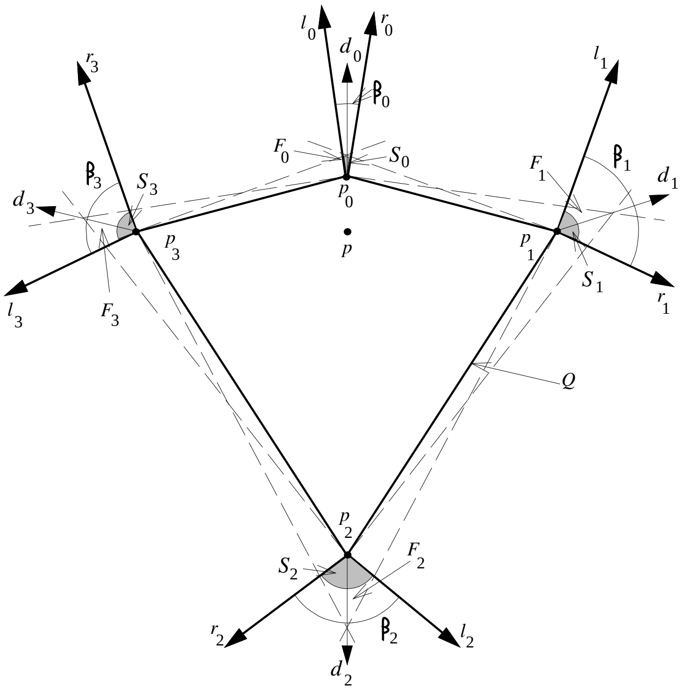

Figure 4: Illustration for Lemma 5.7.

2. the clockwise angle between vector l_i and $\overline{p_i p_{i-1}}$ is greater than $\pi/2$
3. the clockwise angle between vector r_i and $\overline{p_i p_{i+1}}$ is greater than $\pi/2$.

Let d_i be the bisector of angle β_i and refer to $W(p_i, \beta_i, d_i)$ as W_i. Given the bounds on the angles $\angle p_{i-1}p_ip_{i+1}$, we can guarantee that $\beta_1, \beta_2, \beta_3$ are greater than $\pi/2$ and that β_0 is greater than 0. Now, let F_i be the region $W_i \cap HP(r_{i-i}, p_{i-1}) \cap HP(l_{i+1}, p_{i+1})$.

Given a point $x \in F_j$ and a point $y \in F_{j+1}$, we see that $\angle xp_jy > \pi/2$ since $y \in HP(r_j, p_j)$. Similarly, $\angle yp_{j+1}x > \pi/2$ since $x \in HP(l_{j+1}, p_{j+1})$. Similar arguments hold for any non-consecutive pairs of regions F_j and F_k $(j \neq k)$. Thus any sector with apex p_i and angle β_i contained in F_i can serve as a sector S_i. ∎

The preceding lemmas will be used to prove the correctness of a procedure which produces a GG drawing of any non-forbidden tree T.

Lemma 5.8 *Let T be a non-forbidden tree and let $v_0 \in V(T)$ be a leaf of T. Then*

1. *If (T, v_0) is wide, then it can be α-drawn for any $\alpha > \pi/2$.*
2. *If (T, v_0) is not wide then it can be α-drawn for any $\alpha > 0$.*

Proof: (Sketch). We will prove the lemma by induction on the number of vertices in T. The base case, where $|V(T)| = 1$, is obvious. Suppose now that the lemma holds for all trees having fewer than n vertices; let T be a non-forbidden tree with n vertices, let v_0 be a leaf of T and let v be the unique neighbor of v_0.

Assume first that (T, v_0) is wide; given an $\alpha > \pi/2$, we need to describe an α-drawing of (T, v_0). Choose an α' such that $\pi/2 < \alpha' < \alpha$. By Lemma 5.7, the induced subgraph of T consisting of v and its four neighbors v_0, v_1, v_2, v_3 can be drawn as a gabriel tree with $v = p$, $v_i = p_i$ $(0 \leq i \leq 3)$, and $\angle p_3 p_0 p_1 = \alpha'$. Let Q denote the convex quadrilateral determined by the neighbors of v. Since $\alpha' < \alpha$, by Lemma 5.7, we can choose the radii ρ_i of the sectors S_i such that the sectors and the quadrilateral Q lie within an α-sector having p_0 as its apex. Now, for each $1 \leq i \leq 3$, $deg(v_i) \leq 3$. If $deg(v_i) < 3$ then, by induction, rooted tree $(T_{v_i}(v), v_i)$ can be drawn as a gabriel tree in the sector S_i since, in $(T_{v_i}(v), v_i)$, v_i has degree one. So, suppose that $deg(v_i) = 3$ for some $1 \leq i \leq 3$ and let v, a, b be the neighbors of v_i. Since $deg(v) = 4$, $(T_a[v_i], v_i)$ cannot be wide or else T is forbidden; similarly, $(T_b[v_i], v_i)$ cannot be wide. Thus, by induction, each of these rooted trees can be drawn as a gabriel tree in a sector of any angle greater than 0. Partition the sector S_i into 3 sectors, one of angle slightly greater than $\pi/2$ with a sector of some small angle greater than 0 to each side. Now draw $(T_a[v_i], v_i)$ as a gabriel tree in one of the two small sectors and draw $(T_b[v_i], v_i)$ in the other. Since, for any $x \in (T_a[v_i], v_i)$ and $y \in (T_b[v_i], v_i)$, $\angle x v_i y > \pi/2$, the drawing of $(T_{v_i}(v), v_i)$ obtained is a gabriel tree. Thus, the drawing of (T, v_0) obtained is a gabriel tree.

Suppose now that (T, v_0) is not wide; then $2 \leq deg(v) \leq 4$. This time we must show that we can draw (T, v_0) as a gabriel tree in an α-sector for any $\alpha > 0$. Choose α' such that $0 < \alpha' < \alpha$. Assume that $deg(v) = 4$, again with v having neighbors v_0, v_1, v_2, v_3. Since (T, v_0) is not wide, at least one of v_1, v_2, v_3 has degree at most 2; assume, without loss of generality, that this is v_2. Draw the induced subgraph of T consisting of v and its four neighbors v_0, v_1, v_2, v_3 as a gabriel tree by Lemma 5.7, with $v = p$, $v_0 = p_2$, $v_1 = p_1$, $v_2 = p_0$, and $v_3 = p_3$. Let Q denote the convex quadrilateral determined by the neighbors of v. Sectors S_1, S_0, S_3, for v_1, v_2, v_3 respectively, are such that S_1 and S_3 have angles greater than $\pi/2$ and S_2 has an angle greater than 0, by Lemma 5.7. Since $\alpha' < \alpha$, we can choose the radii ρ_i of the sectors S_1, S_0, S_3 such that the sectors and the quadrilateral Q lie within an α-sector having v_0 as its apex. Using induction as above, each of the rooted trees $(T_{v_i}(v), v_i)$ can be drawn as a gabriel tree in the sector with apex v_i in such a way that the drawing of (T, v_0) obtained is a gabriel tree.

We now address the cases where $deg(v) = 2$ or $deg(v) = 3$. In the former case, let x be the neighbor of v other than v_0, by induction $(T_x[v], v)$ can be drawn in an α-sector for any $\alpha > \pi/2$. By drawing v_0 on the line determined by vx suitably far away from $(T_x[v], v)$, the tree (T, v_0) can be drawn in a sector of arbitrarily small angle. In the latter case, denoting by x and y the neighbors of v other than v_0, and by noting that neither $(T_x[v], v)$ nor $(T_y[v], v)$ can be wide (or else T is forbidden), we can, as above, take any sector of angle greater than $\pi/2$, partition it into three sectors, one of angle greater than $\pi/2$ bordered by two angles each greater than 0, and draw $(T_x[v], v)$ and $(T_y[v], v)$ in the smaller sectors. This gives a drawing of $(T_v(v_0), v)$ as a gabriel tree in a sector of angle slightly greater than $\pi/2$ which we can extend to a gabriel drawing of (T, v_0) in a sector of arbitrarily small angle by drawing v_0 far enough away from $(T_v(v_0), v)$. ■

The drawing algorithm implicit in the preceding lemma takes time proportional to the size of the tree using the real RAM model of computation. The tree of Figure 4 is produced by such algorithm.

Thus Lemmas 5.6 and 5.8 together form the proof of the following theorem.

Theorem 5.2 *A tree T is a gabriel tree if and only if it is not forbidden. Furthermore, in the real RAM model, it is possible to compute a GG drawing of T in time proportional to the size of T.*

6 Conclusions

In this paper we have provided complete characterizations of trees that can be drawn as relatively closest graphs, relative neighborhood graphs, gabriel graphs or modified gabriel graphs. The characterizations give rise to linear time recognition and drawing algorithms for each type of proximity graph. The problems of completely characterizing which graphs admit RNG, RCG, MGG or GG drawings remain open. Even characterizing those planar graph which admit such drawings appears to be difficult.

Our algorithms produce drawings with the following property: For each pair of non-adjacent vertices u and v of the tree T, the region of influence of u and v contains at least one vertex of the (unique) path joining u and v. We call such drawings *regular* drawings. Which graphs admit a regular RNG (RCG, MGG, GG) drawing?

The model of computation used is the real RAM. Although the decision algorithms do not need such a strong model of computation, the drawing algorithms do, since the coordinates for the vertices are real numbers. Suppose we want to find an RNG drawing of a path P of n vertices by means of the algorithm presented in Section 4. Depending on the sizes of the radii of the sectors in which the subtrees are drawn, the ratio between the shortest edge and the longest edge in the drawing may be $O(k^n)$ for some $k > 1$. This leads to a drawing of exponential area. Of course, there exist alternative drawings for P where the edge lengths are all equal. Is it possible to provide drawing algorithms which require area polynomial in the size of the tree?

Acknowledgments We thank G.Di Battista, G.Toussaint, and S.Whitesides for their comments and suggestions. We are particularly grateful to G.Blanco and J.Garcia for the interesting discussions that produced the ideas underlying Theorem 5.1; also, they showed us a counter example to Sendov's theorem. Finally, we thank David Thompson and Al Van Houtte for providing a stimulating research environment.

References

[1] P. K. Agarwal and J. Matoušek. Relative Neighbourhood Graphs in Three Dimensions. *Computational Geometry: Theory and Applications,* **2**, 1992, pp. 1-14.

[2] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications*. Elsevier Science, New York, New York, 1976.

[3] P. Bose, W. Lenhart, and G. Liotta. Characterizing Proximity Trees. Technical Report no. TR-SOCS 93.9, School of Computer Science, McGill University, 1993.

[4] R. J. Cimikowski. Properties of Some Euclidean Proximity Graphs. *Pattern Recognition Letters,* **13**, 1992, pp. 417-423.

[5] B. Chazelle, H. Edelsbrunner, L. J. Guibas, J. E. Hershberger, R. Seidel, and M. Sharir. Slimming Down by Adding; Selecting Heavily Covered Points. *Proc. ACM Symp. on Comp. Geom.*, 1990, pp. 116-127.

[6] G. Di Battista, P. Eades, R. Tamassia and I.G. Tollis. Algorithms for Automatic Graph Drawing: An Annotated Bibliography. To appear in *Computational Geometry: Theory and Applications*.

[7] G. Di Battista and L. Vismara. Angles of Planar Triangular Graphs. *Proc. STOC '93,* 1993, pp. 431-437.

[8] M. B. Dillencourt and W. D. Smith. Graph-Theoretical Conditions for Inscribability and Delaunay Realizability. Univ. of California Irvine, TR 92-90.

[9] P. Eades and S. Whitesides. The Realization Problem for Euclidean Minimum Spanning Trees is NP–hard. To appear in *Proc. ACM Symp. on Comp. Geom.*, 1994.

[10] H. Edelsbrunner and T. S. Tan. A Quadratic Time Algorithm for the MinMax Length Triangulation. *Proc. FOCS '91*, 1991, pp. 414-423.

[11] H. Edelsbrunner, D. G. Kirkpatrick, and R. Seidel. On the Shape of a Set of Points in the Plane. *IEEE Transaction on Information Theory,* **29**, 1983, pp. 551-559.

[12] K. R. Gabriel and R. R. Sokal. A New Statistical Approach to Geographical Analysis. *Systematic Zoology,* **18**, 1969, pp. 54-64.

[13] J. Garcia. A Counter-Example to a Theorem of Sendov. Manuscript in preparation.

[14] D. G. Kirkpatrick and J. D. Radke. A Framework for Computational Morphology. *Computational Geometry*, ed. G. T. Toussaint, Elsevier, Amsterdam, 1985, pp. 217-248.

[15] M. S. Jacobson, M. J. Lipman, and F. R. McMorris. Trees that are Sphere-of-Influence Graphs. Univ. of Louisville, TR 0191-2.

[16] J. W. Jaromczyk and G. T. Toussaint. Relative Neighborhood Graphs and Their Relatives. *Proceedings of the IEEE,* **80**, 1992, pp. 1502-1517.

[17] P. M. Lankford. Regionalization: Theory and Alternative Algorithms. *Geographical Analysis,* **1**, 1969, pp. 196-212.

[18] A. Lubiw, and N. Sleumer, All Maximal Outerplanar Graphs are Relative Neighborhood Graphs. *Proc. CCCG '93*, 1993, pp. 198-203.

[19] D. W. Matula and R. R. Sokal. Properties of Gabriel Graphs Relevant to Geographic Variation Research and the Clustering of Points in the Plane. *Geographical Analysis,* **12**, 1980, pp. 205-222.

[20] C. Monma and S. Suri. Transitions in Geometric Minimum Spanning Trees. *Proc. ACM Symp. on Comp. Geom.*, 1991, pp. 239-249.

[21] M. S. Paterson and F.F. Yao, On Nearest-Neighbor Graphs. *Proc. ICALP '92*, 1992, pp. 416-426.

[22] F. P. Preparata and M. I. Shamos, *Computational Geometry – an Introduction.* Springer-Verlag, New York, 1985.

[23] J. D. Radke. On the shape of a set of points. *Computational Morphology,* ed. G. T. Toussaint, Elsevier, Amsterdam, 1988, pp. 105-136.

[24] B. Sendov. Planar Neighbourhood Graphs without Cycles. *Comptes Rendus de l'Academie Bulgare des Sciences,* **44**, 1991, pp. 23-25.

[25] K. J. Supowit. The Relative Neighborhood Graph, with an Application to Minimum Spanning Trees. *Journal of the Association of Computing Machinery,* **30**, 1983, pp. 429-448.

[26] G. T. Toussaint. The Relative Neighborhood Graph of a Finite Planar Set. *Pattern Recognition,* **12**, 1980, pp. 261-268.

[27] G. T. Toussaint. Pattern Recognition and Geometrical Complexity *Proc. Fifth Int. Conf. on Pattern Recognition* 1980, pp. 1340-1347.

[28] G. T. Toussaint. A Graph-Theoretical Primal Sketch. *Computational Morphology*, ed. G. T. Toussaint, Elsevier, Amsterdam, 1988, pp. 229-260.

[29] G. T. Toussaint. Some Unsolved Problems on Proximity Graphs. *Proc. Workshop on Proximity Graphs*, 1991.

[30] R. B. Urquhart. Some Properties of the Planar Euclidean Relative Neighbourhood Graph. *Pattern Recognition Letters,* **1**, 1983, pp. 317-322.

[31] R. C. Vetkamp. The γ-Neighbourhood Graph. *Computational Geometry: Theory and Applications,* **1**, 1992, pp. 227-246.