Online Routing in Triangulations*

Prosenjit Bose and Pat Morin

School of Computer Science, Carleton University
Ottawa, Canada, K1S 5B6
{jit,morin}@scs.carleton.ca

Abstract. We consider online routing strategies for routing between the vertices of embedded planar straight line graphs. Our results include (1) two deterministic memoryless routing strategies, one that works for all Delaunay triangulations and the other that works for all regular triangulations, (2) a randomized memoryless strategy that works for all triangulations, (3) an $O(1)$ memory strategy that works for all convex subdivisions, (4) an $O(1)$ memory strategy that approximates the shortest path in Delaunay triangulations, and (5) theoretical and experimental results on the competitiveness of these strategies.

1 Introduction

In this paper we consider online routing in the following abstract setting: The environment is a planar straight line graph [12], T, with n vertices, whose edges are weighted by the Euclidean distance between their endpoints, the source v_{src} and destination v_{dst} are vertices of T, and a packet can only move on edges of T. Initially, a packet only knows v_{src}, v_{dst}, and $N(v_{src})$, where $N(v)$ denotes the set of vertices adjacent to v.

We classify online routing strategies based on their use of memory and/or randomization. Define v_{cur} as the vertex at which the packet is currently stored. A routing strategy is called *memoryless* if the next step taken by a packet depends only on v_{cur}, v_{dst}, and $N(v_{cur})$. A strategy is *randomized* if the next step taken by a packet is chosen randomly from $N(v_{cur})$. A randomized strategy is memoryless if the distribution used to choose from $N(v_{cur})$ is a function only of v_{cur}, v_{dst}, and $N(v_{cur})$.

For a strategy $\mathcal{S}$ we say that a graph *defeats* $\mathcal{S}$ if there is a source/destination pair such that a packet never reaches the destination when beginning at the source. If $\mathcal{S}$ finds a path P from v_{src} to v_{dst} we call P the $\mathcal{S}$ path from v_{src} to v_{dst}. Here we use the term *path* in an intuitive sense rather than a strict graph theoretic sense, since P may visit the same vertex more than once.

In this paper we also consider, as a special case, a class of "well-behaved" triangulations. The *Voronoi diagram* [11] of S is a partitioning of space into cells such that all points within a Voronoi cell are closer to the same element $p \in S$

* This research was supported by the Natural Sciences and Engineering Research Council of Canada.

than any other point in S. The *Delaunay triangulation* is the straight-line face dual of the Voronoi diagram, i.e., two points in S have an edge between them in the Delaunay triangulation if their Voronoi regions have an edge in common.

In this paper we consider several different routing strategies and compare their performance empirically. In particular, we describe (1) a memoryless strategy that is not defeated by any Delaunay triangulation, (2) a memoryless strategy that is not defeated by any regular triangulation, (3) a memoryless randomized strategy that uses 1 random bit per step and is not defeated by any triangulation, (4) a strategy that uses $O(1)$ memory that is not defeated by any convex subdivision, (5) a strategy for Delaunay triangulations that uses $O(1)$ memory in which a packet never travels more than a constant times the Euclidean distance between v_{src} and v_{dst}, and (6) a theoretical and empirical study of the quality (length) of the paths found by these strategies.

The first four routing strategies are described in Section 2. Section 3 presents theoretical and empirical results on the length of the paths found by these strategies and describes our strategy for Delaunay triangulations. A discussion of related previous work is provided in Section 4. Finally, Section 5 summarizes and describes directions for future research.

Due to space constraints, some proofs and figures are omitted from this extended abstract. The interested reader is referred to the full version of the paper [3].

2 Four Simple Strategies

In this section we describe four online routing strategies and prove theorems about which types of graphs never defeat them. We begin with the simplest (memoryless) strategies and proceed to the more complex strategies.

2.1 Greedy Routing

The *greedy routing* (GR) strategy always moves the packet to the neighbor $gdy(v_{cur})$ of v_{cur} that minimizes $dist(gdy(v_{cur}), v_{dst})$, where $dist(p, q)$ denotes the Euclidean distance between p and q. In the case of ties, one of the vertices is chosen arbitrarily. The greedy routing strategy can be defeated by a triangulation T in two ways: (1) the packet can get trapped moving back and forth on an edge of the triangulation (Fig. 1 (a)), or (2) the packet can get trapped on a cycle of three or more vertices (Fig. 1 (b)). However, as the following theorem shows, neither of these situations can occur if T is a Delaunay triangulation.

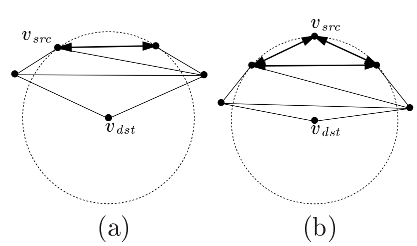


Fig. 1. Triangulations that defeat the greedy routing strategy.

Theorem 1 *There is no point set whose Delaunay triangulation defeats the greedy routing strategy.*

2.2 Compass Routing

The *compass routing* (CR) strategy always moves the packet to the vertex $cmp(v_{cur})$ that minimizes the angle $\angle v_{dst}, v_{cur}, cmp(v_{cur})$ over all vertices adjacent to v_{cur}. Here the angle is taken to be the smaller of the two angles as measured in the clockwise and counterclockwise directions. In the case of ties, one of the (at most 2) vertices is chosen using some arbitrary deterministic rule.

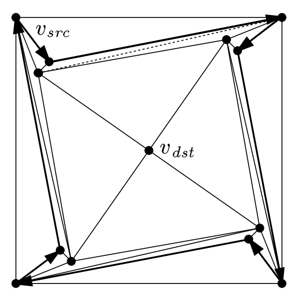


Fig. 2. A triangulation that defeats the compass routing strategy.

One might initially believe (as we did) that compass routing can always be used to find a path between any two vertices in a triangulation. However, the triangulation in Fig. 2 defeats compass routing. When starting from one of the vertices on the outer face of T, and routing to v_{dst}, the compass routing strategy gets trapped on the cycle shown in bold. The following lemma shows that any triangulation that defeats compass routing causes the packet to get trapped in a cycle.

Lemma 1 *Let T be a triangulation that defeats compass routing, and let v_{dst} be a vertex such that compass routing fails to route a packet to v_{dst} when given some other vertex as the source. Then there exists a cycle $C = v_0, \ldots, v_{k-1}$ ($k \geq 3$) in T such that $cmp(v_i) = v_{i+1}$ for all $0 \leq i < k$.*[1]

We call such a cycle, C, a *trapping cycle* in T for v_{dst}. Next we characterize trapping cycles in terms of a visibility property of triangulations. Let t_1 and t_2 be two triangles in T. Then we say that t_1 *obscures* t_2 if there exists a ray originating at v_{dst} that strikes t_1 first and then t_2. Let u and v be any two vertices of T such that $cmp(u) = v$. Then define $\triangle uv$ as the triangle of T that is contained in the closed half-plane bounded by the line through uv and that contains v_{dst}. We obtain the following useful characterization of trapping cycles.

Lemma 2 *Let T be a triangulation that defeats compass routing and let $C = v_0, \ldots, v_{k-1}$ be a trapping cycle in T for vertex v_{dst}. Then $\triangle v_i v_{i+1}$ is either identical to, or obscures $\triangle v_{i-1} v_i$, for all $0 \leq i < k$.*

A *regular triangulation* [13] is a triangulation obtained by orthogonal projection of the faces of the lower hull of a 3-dimensional polytope onto the plane. Note that the Delaunay triangulation is a special case of a regular triangulation in which the vertices of the polytope all lie on a paraboloid. Edelsbrunner [7] showed that if T is a regular triangulation, then T has no set of triangles that

[1] Here and henceforth, all subscripts are assumed to be taken mod$k$.

obscure each other cyclically from *any* viewpoint. This result, combined with Lemma 2, yields our main result on compass routing.

Theorem 2 *There is no regular triangulation that defeats the compass routing strategy.*

2.3 Randomized Compass Routing

In this section, we consider a randomized routing strategy that is not defeated by any triangulation. Let $cw(v)$ be the vertex in $N(v)$ that minimizes the clockwise angle $\angle v_{dst}, v, cw(v)$ and let $ccw(v)$ be the vertex in $N(v)$ that minimizes the counterclockwise angle $\angle v_{dst}, v, ccw(v)$. Then the *randomized compass routing* (RCR) strategy moves the packet to one of $\{cw(v_{cur}), ccw(v_{cur})\}$ with equal probability.

Before we can make statements about which triangulations defeat randomized compass routing, we must define what it means for a triangulation to defeat a randomized strategy. We say that a triangulation T defeats a (randomized) routing strategy if there exists a pair of vertices v_{src} and v_{dst} of T such that a packet originating at v_{src} with destination v_{dst} has probability 0 of reaching v_{dst} in any finite number of steps.

Note that, since randomized compass routing is memoryless, proving that a triangulation T does not defeat randomized compass routing implies that a packet reaches its destination with probability 1. The following theorem shows the versatility of randomized compass routing.

Theorem 3 *There is no triangulation that defeats the randomized compass routing strategy.*

Proof. Assume, by way of contradiction that a triangulation T exists that defeats the randomized compass routing strategy. Then there is a vertex v_{dst} of T and a minimal set S of vertices such that: (1) $v_{dst} \notin S$, (2) the subgraph H of T induced by S is connected, and (3) for every $v \in S$, $cw(v) \in S$ and $ccw(v) \in S$.

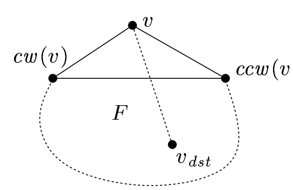


Fig. 3. The proof of Theorem 3.

Refer to Fig. 3 for what follows. The vertex v_{dst} lies in some face F of H. Let v be a vertex on the boundary of F such that the line segment (v, v_{dst}) is contained in F. Such a vertex is guaranteed to exist [5]. The two neighbours of v on the boundary of F must be $cw(v)$ and $ccw(v)$ and these cannot be the same vertex (since F contains (v, v_{dst}) in its interior). Note that, by the definition of $cw(v)$ and $ccw(v)$, and by the fact that T is a triangulation, the triangle $(cw(v), v, ccw(v))$ is in T. But this is a contradiction, since then v is not on the boundary of F. $\square$

2.4 Right-Hand Routing

The folklore "right-hand rule" for exploring a maze states that if a player in a maze walks around never lifting her right-hand from the wall, then she will eventually visit every wall in the maze. More specifically, if the maze is the face of a connected planar straight line graph, the player will visit every edge and vertex of the face [2].

Let T be any convex subdivision. Consider the planar subdivision T' obtained by deleting from T all edges that properly intersect the line segment joining v_{src} and v_{dst}. Because of convexity, T' is connected, and v_{src} and v_{dst} are on the boundary of the same face F of T'. The *right-hand routing* (RHR) strategy uses the right-hand rule on the face F to route from v_{src} to v_{dst}. Right-hand routing is easily implemented using only $O(1)$ additional memory by remembering v_{src}, v_{dst}, and the last vertex visited.

Theorem 4 *There is no convex subdivision that defeats the right-hand routing strategy.*

3 Competitiveness of Paths

Thus far we have considered only the question of whether routing strategies can find *a* path between any two vertices in T. An obvious direction for research is to consider the length of the path found by a routing strategy. We say that a routing strategy is c-competitive for T if for any pair (v_{src}, v_{dst}) in T, the length (sum of the edge lengths) of the path between v_{src} and v_{dst} found by the strategy is at most c times the length of the shortest path between v_{src} and v_{dst} in T. In the case of randomized strategies, we use the expected length of the path. A strategy has a competitive ratio of c if it is c-competitive.

This section addresses questions about the competitive ratio of the strategies described so far, as well as a new strategy specifically targeted for Delaunay triangulations. We present theoretical as well as experimental results.

3.1 Negative Results

It is not difficult to contrive triangulations for which none of our strategies are c-competitive for any constant c. Thus it is natural to restrict our attention to a well behaved class of triangulations. Unfortunately, even for Delaunay triangulations none of the strategies described so far are c-competitive.

Theorem 5 *There exists Delaunay triangulations for which none of the greedy, compass, randomized compass, or right-hand routing strategies are c-competitive for any constant c.*

3.2 A c-Competitive Strategy for Delaunay Triangulations

Since none of the strategies described in Section 2 is competitive, even for Delaunay triangulations, an obvious question is whether there exists any strategy that is competitive for Delaunay triangulations. In this section we answer this question in the affirmative. Our strategy is based on the remarkable proof of Dobkin *et al* [6] that the Delaunay triangulation approximates the complete Euclidean graph to within a constant factor in terms of shortest path length. In the following we will use the notation $\mathrm{x}(p)$ (resp. $\mathrm{y}(p)$) to denote the x-coordinate (resp. y-coordinate) of the point p, and the notation $|X|$ to denote the Euclidean length of the path X.

Consider the directed line segment from v_{src} to v_{dst}. This segment intersects regions of the Voronoi diagram in some order, say $R_0, \ldots, R_{m-1}$, where R_0 is the Voronoi region of v_{src} and R_{m-1} is the Voronoi region of v_{dst}. The *Voronoi routing* (VR) strategy for Delaunay triangulations moves the packet from v_{src} to v_{dst} along the path $v_0, \ldots, v_{m-1}$ where v_i is the site defining R_i. An example of a path obtained by the Voronoi routing strategy is shown in Fig. 4. Since the Voronoi region of a vertex v can be computed given only the neighbours of v in the Delaunay triangulation, it follows that the Voronoi routing strategy is an $O(1)$ memory routing strategy.

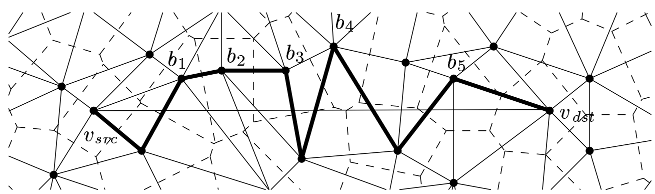


Fig. 4. A path obtained by the Voronoi routing strategy.

The Voronoi routing strategy on its own is not c-competitive for all Delaunay triangulations. However, it does have some properties that allow us to derive a c-competitive strategy. As with right-hand routing, let T' be the graph obtained from T by removing all edges of T that properly intersect the segment (v_{src}, v_{dst}), and let F be the face of T' that contains both v_{src} and v_{dst}. Assume wlog that v_{src} and v_{dst} both lie on the x-axis and that $\mathrm{x}(v_{src}) < \mathrm{x}(v_{dst})$. The following results follow from the work of Dobkin *et al* [6].

Lemma 3 *The Voronoi path is x-monotone, i.e., $\mathrm{x}(v_i) < \mathrm{x}(v_j)$ for all $i < j$.*

Lemma 4 *Let P' be the collection of maximal subpaths of $v_0, \ldots, v_{m-1}$ that remain above the x-axis, i.e., $P' = \{v_i, \ldots, v_j : \mathrm{y}(v_{i-1}) < 0$ and $\mathrm{y}(v_{j+1}) < 0$ and $\mathrm{y}(v_k) \geq 0$ for all $i \leq k \leq j\}$. Then $\sum_{X \in P'} |X| \leq (\pi/2) dist(v_{src}, v_{dst})$.*

Lemma 5 *Select any i such that $y(v_i) \geq 0$ and $\mathrm{y}(v_{i+1}) < 0$. Let $j > i$ be the least value such that $\mathrm{y}(v_j) \geq 0$ and let $c_{dfs} = (1+\sqrt{5})\frac{\pi}{2}$.[2] Then the length of the*

[2] We call c_{dfs} the Dobkin, Friedman and Supowit constant [6].

path $v_i, \ldots, v_j$ *is at most* $c_{dfs}(\mathrm{x}(v_j) - \mathrm{x}(v_i))$ *or the length of the upper boundary of* F *between* v_i *and* v_j *is at most* $c_{dfs}(\mathrm{x}(v_j) - \mathrm{x}(v_i))$.

Our c-competitive routing strategy will visit the subsequence $b_0, \ldots, b_{l-1}$ of vertices $v_0, \ldots, v_{m-1}$ that are above or on the segment (v_{src}, v_{dst}). (Refer to Fig. 4.) If b_i and b_{i+1} are consecutive on the Voronoi path (i.e., $b_i = v_j$ and $b_{i+1} = v_{j+1}$ for some j) then our strategy will use the Voronoi path (i.e, the direct edge) from b_i to b_{i+1}. On the other hand, if b_i and b_{i+1} are not consecutive on the Voronoi path then by Lemma 5, there exists a path from b_i to b_{i+1} of length at most $c_{dfs}(\mathrm{x}(b_{i+1}) - \mathrm{x}(b_i))$.

The difficulty occurs because the strategy does not know beforehand which path to take. The solution is to simulate exploring both paths "in parallel" and stopping when the first one reaches b_{i+1}.[3]

More formally, let $P_V = (p_0 = b_i, \ldots, p_n = b_{i+1})$ be the Voronoi path from b_i to b_{i+1} and let $P_F = (q_0 = b_i, \ldots, q_n = b_{i+1})$ be the path from b_i to b_{i+1} on the upper boundary of F. The strategy is described by the following algorithm.

$j \leftarrow 0$, $l_0 \leftarrow \min\{dist(p_0, p_1), dist(q_0, q_1)\}$.
repeat
  Explore P_F until reaching b_{i+1} or until reaching a vertex q_x such that $|q_0, \ldots, q_{x+1}| > 2l_j$. If b_{i+1} is reached quit, otherwise return to b_i.
  $j \leftarrow j + 1$, $l_j \leftarrow |q_0, \ldots, q_{y+1}|$.
  Explore P_V until reaching b_{i+1} or until reaching a vertex p_y such that $|p_0, \ldots, p_{y+1}| > 2l_j$. If b_{i+1} is reached then quit, otherwise return to b_i.
  $j \leftarrow j + 1$, $l_j \leftarrow |p_0, \ldots, p_{y+1}|$.
until b_{i+1} is reached

Lemma 6 *Using the parallel search strategy described above, a packet reaches* b_{i+1} *after traveling a distance of at most* $9c_{dfs}(\mathrm{x}(b_{i+1}) - \mathrm{x}(b_i)) \sim 45.75(\mathrm{x}(b_{i+1}) - \mathrm{x}(b_i))$.

Given the positions of v_{dst} and v_{src} the parallel search strategy described above is easily implemented as part of an $O(1)$ memory routing strategy. We refer to the combination of the Voronoi routing strategy with the parallel search strategy described above as the *parallel Voronoi routing* (PVR) strategy.

Theorem 6 *The parallel Voronoi routing strategy is* $(9c_{dfs} + \pi/2)$*-competitive for all Delaunay triangulations.*

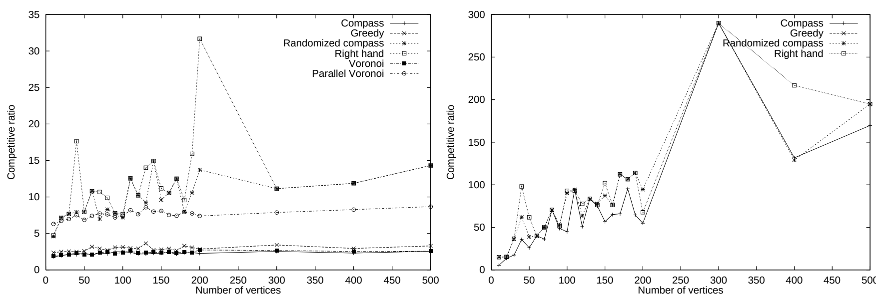
Proof. The strategy incurs two costs: (1) the cost of traveling on subpaths of the Voronoi path that remain above the y-axis, and (2) the cost of applications of the parallel search strategy. By Lemma 4, the first cost is at most $(\pi/2) \cdot dist(v_{src}, v_{dst})$. By Lemma 6 and the fact that $b_0, \ldots, b_{l-1}$ is x-monotone (Lemma 3), the cost of the second is at most $9c_{dfs} \cdot dist(v_{src}, v_{dst})$. Since the Euclidean distance between two vertices of T is certainly a lower bound on their shortest path distance in T the theorem follows. □

[3] A similar strategy for finding an unknown target point on a line is given by Baeza-Yates *et al* [1]. See also Klein [8].

3.3 Empirical Results

While it is sometimes possible to come up with pathological examples of triangulations for which a strategy is not competitive, it is often more reasonable to use the competitive ratio of a strategy on average or random inputs as an indicator of how it will perform in practice. In this section we describe some experimental results about the competitiveness of our strategies. All experiments were performed on sets of points randomly distributed in the unit square, and each data point is the maximum of 50 independent trials.

The first set of experiments, shown in Fig. 5 (a), involved measuring the performance of all six routing strategies on Delaunay triangulations. Compass routing, greedy routing, and Voronoi routing consistently achieve better competitive ratios, with greedy routing slightly worse than the other two. Randomized



(a) (b)

Fig. 5. Empirical competitive ratios for (a) Delaunay triangulations and (b) Graham triangulations.

compass routing, right-hand routing, and parallel Voronoi routing had significantly higher competitive ratios. The results for randomized compass routing and right-hand routing show a significant amount of jitter. This is due to the fact that relatively simple configurations[4] that can easily occur in random point sets, result in high competitive ratios for these strategies. On the other hand, parallel Voronoi routing seems much more stable, and achieves better competitive ratios in practice than its worst case analysis would indicate.

The most important conclusion drawn from these experiments is that there are no simple configurations (i.e., that occur often in random point sets) that result in extremely high competitive ratios for greedy, compass, Voronoi, or parallel Voronoi routing in Delaunay triangulations. This suggests that any of these strategies would work well in practice.

The four simple routing strategies of Section 2 were also tested on Graham triangulations. These are obtained by first sorting the points by x-coordinate and then triangulating the resulting monotone chain using a linear time algorithm for computing the convex hull of a monotone polygonal chain [12]. The

[4] These configurations come up in the proof of Theorem 5 [3].

results are shown in Fig. 5 (b). In these tests it was always the case that at least one of the 50 independent triangulations defeated greedy routing. Thus, there are no results shown for greedy routing. The relative performance of the compass, randomized compass, and right hand routing strategies was the same as for Delaunay triangulations. However, unlike the results for Delaunay triangulations, the competitive ratio appears to be increasing linearly with the number of vertices.

4 Comparison with Related Work

In this section we survey related work in the area of geometric online routing, and compare our results with this work. Due to space constraints, we can not provide a comprehensive list of references in this version of the paper. A more complete survey is available in the full version of the paper [3].

Kranakis *et al* [9] study compass routing, and provide a proof that no Delaunay triangulation defeats compass routing. The current paper makes use of a very different proof technique to show that compass routing works for a larger class of triangulations. They also describes an $O(1)$ memory routing strategy that is not defeated by any connected planar graph, thus proving a stronger result than Theorem 4.

Lin and Stojmenović [10] and Bose *et al* [4] consider online routing in the context of *ad hoc* wireless networks modeled by unit disk graphs. They provide simulation results for a variety of strategies that measuring success rates (how often a packet never reaches its destination) as well as hop-counts of these strategies on unit graphs of random point sets.

To the best of our knowledge, no literature currently exists on the competitiveness of geometric routing strategies in our abstract setting, and our parallel Voronoi routing strategy is the first theoretical result in this area.

5 Conclusions

We have studied the problem of online routing in geometric graphs. Our theoretical results show which types of graphs our strategies are guaranteed to work on, while our simulation results rank the performance of the strategies on two types of random triangulations. These results are summarized in the following table.

Strategy	Memory	Randomized	Class of graphs	Rank 1	Rank 2	Competitive
GR	None	No	Delaunay $\triangle$'s	3	–	No
CR	None	No	Regular $\triangle$'s	1	1	No
RCR	None	Yes	All $\triangle$'s	5	2	No
RHR	$O(1)$	No	Convex subd.	6	3	No
VR	$O(1)$	No	Delaunay $\triangle$'s	1	–	No
PVR	$O(1)$	No	Delaunay $\triangle$'s	4	–	Yes

Acknowledgement

The authors would like to thank Silvia Götz for reading and commenting on an earlier version of this paper.

References

[1] R. Baeza-Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Information and Computation*, 106(2):234–252, 1993.

[2] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications*. Elsevier North-Holland, 1976.

[3] P. Bose and P. Morin. Online routing in triangulations. Available online at `http://www.scs.carleton.ca/~morin`, 1999.

[4] P. Bose, P. Morin, I. Stojmenović, and J. Urrutia. Routing with guaranteed delivery in *ad hoc* wireless networks. In *Proceedings of Discrete Algorithms and Methods for Mobility (DIALM'99)*, 1999. To appear.

[5] V. Chvátal. A combinatorial theorem in plane geometry. *Journal of Combinatorial Theory B*, 18:39–41, 1975.

[6] D. Dobkin, S. J. Friedman, and K. J. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete and Computational Geometry*, 5:399–407, 1990. See also *28th Symp. Found. Comp. Sci.*, 1987, pp. 20–26.

[7] H. Edelsbrunner. An acyclicity theorem for cell complexes in d dimension. *Combinatorica*, 10(3):251–260, 1988.

[8] R. Klein. Walking an unknown street with bounded detour. *Computational Geometry Theory and Applications*, 1:325–351, 1992.

[9] E. Kranakis, H. Singh, and J. Urrutia. Compass routing on geometric networks. In *Proceedings of the 11th Canadian Conference on Computational Geometry (CCCG'99)*, 1999. To appear.

[10] X. Lin and I. Stojmenović. Geographic distance routing in ad hoc wireless networks. Technical Report TR-98-10, SITE, University of Ottawa, December 1998.

[11] A. Okabe, B. Boots, and K. Sugihara. *Spatial Tesselations: Concepts and Applications of Voronoi Diagrams*. John Wiley and Sons, 1992.

[12] Franco P. Preparata and Michael Ian Shamos. *Computational Geometry*. Springer-Verlag, New York, 1985.

[13] Günter M. Ziegler. *Lectures on Polytopes*. Number 154 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1994.