Geometric and Computational Aspects of Manufacturing Processes

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Abstract

A survey is presented on the geometric and computational aspects of injection molding, gravity casting, stereolithography and laser sculpting. Emphasis is placed on the two fundamental questions that arise concerning every type of manufacturing process:

1. Given an object, can it be constructed using a particular process?

2. Given that an object can be built using a particular process, what is the *best* way to construct the object?

1 Introduction

In the manufacturing industry, there are many different processes such as molding, casting, NC machining, laser sculpting, automated welding and 3-D printing (stereolithography), available to construct an object. However, every manufacturing process imposes certain restrictions on the types of objects that can be constructed as well as the way a given object may be built. For example, a sphere cannot be built using stereolithography, but can be easily built using injection molding. Also, a cube can be constructed using stereolithography only when placed on one of its faces. Two fundamental questions arise concerning every type of manufacturing process:

1. Given an object, can it be constructed using a particular process?
2. Given that an object can be built using a particular process, what is the *best* way to construct the object?

The latter question gives rise to many different problems depending on how *best* is qualified. The geometry of the object, coupled with the restrictions imposed by the particular manufacturing process under consideration, play a vital role in determining the answer to these questions.

The importance of these questions is quite evident. For example, when designing an object to be built by a certain type of manufacturing process, currently an engineer must always keep in mind the process that is to be used to construct the object. This limits the creativity of the engineer since the question of design feasibility must always be kept in mind while creating the object. In fact, the engineer is never really quite sure whether the object can be built since no formal methods exist to determine the feasibility of an object for most manufacturing processes. To resolve this problem, practical algorithms are needed to determine, given an arbitrary object, whether or not it can be built using any of the known manufacturing processes. The benefits of such a CAD/CAM system would be two-fold. Firstly, an engineer would have an algorithm to verify whether an object can be created using the manufacturing process intended (i.e. a type of automatic design verification). Secondly, a list of the possible manufacturing processes that can build a particular object would allow an engineer to design something and then determine which manufacturing process would be most cost efficient.

In the last two years, there has been a flurry of activity concerning the study of manufacturing processes from this perspective. In this paper, we survey the recent developments on the geometric and computational aspects of injection molding, gravity casting, laser or drill sculpting and stereolithography.

2 Molding in Two Dimensions

'During injection, the mold is tilted into a favorable position that will eliminate surface defects such as bubbles and insure a complete fill. Mold orientation during fill is a cut-and-try process to find the most favorable position.' [Pri87]

The main motive for focusing on the geometry of the object to be molded, has been to remove the 'cut-and-try' from the process of finding a favorable mold position. A mold, as defined in [Bo79], refers to the whole assembly of parts that make up a cavity into which liquid is poured to give the shape of the required component when the liquid hardens. Given a mold, establishing whether there exists an orientation that allows the filling of the mold using only one pin gate (the pin gate is the point from which the liquid is injected into the mold) as well as determining an orientation that allows the most complete fill are two major problems in the field of injection molding.

These problems are difficult when the focus is on the fluid dynamics and physics of the whole molding process. To date, only heuristics have been proposed as solutions to these two general problems ([Pri87], [Mac89], [Bo79], [HT92]) because of the complexity involved. However, when viewed from a purely geometric perspective, optimal solutions exist. In this section, we review the techniques developed in the two-dimensional case (where the mold is modelled as a polygon) which find applications to polymer molds. In practice many 3-dimensional objects are almost flat so that in effect they can be considered as 2-dimensional. Therefore the 2-dimensional theory is more important than may appear at first glance.

2.1 Notation and Preliminaries

A *simple polygon* P is a simply connected subset of the plane whose boundary is a closed chain of line segments. We will represent a polygon, P, by a set of vertices $v_1, v_2, \ldots, v_{n-1}, v_n$ such that each pair of consecutive vertices is joined by an edge, including the pair $\{v_n, v_1\}$. We assume that the vertices are in clockwise order, so that the interior of the polygon lies to the right as the boundary of the polygon is traversed.

We will denote the open interior of the polygon P by $int(P)$, the boundary by $bd(P)$, and the open exterior by $ext(P)$. The boundary is considered part of the polygon; that is, $P = int(P) \cup bd(P)$. We assume that the vertices of the polygon are in general position, i.e., no three vertices are collinear. This assumption simplifies discussion but is not an inherent limitation of the methods described.

We assume that the liquid (or molten metal) begin poured into the mold has small viscosity and small surface tension much like water. The point on the polygon boundary from which the liquid is poured into the polygon is called the *pin gate*. We assume that the pin gate is the only point from which air is allowed to escape unless stated otherwise. A *venting hole* is a point from which only air and no liquid is allowed to escape. We assume that neither the liquid being poured into the mold, nor the air in the mold are compressible. Finally, we assume that air cannot bubble out through the liquid. This geometric model is called the *gravity casting* model.

When liquid is poured into a polygon, the level of the liquid rises in the direction opposite that of gravity. The lowest horizontal line such that all the liquid in the polygon is contained below it, is defined as the *level line*. It is possible for the level line to be higher than the level of the liquid in some section of the polygonal mold. For example, the situation depicted in Figure 1 can occur while the mold is being injected with liquid. Thus we define a *level chord* to be the horizontal chord representing the level of liquid in the sub-polygon lying below the chord.

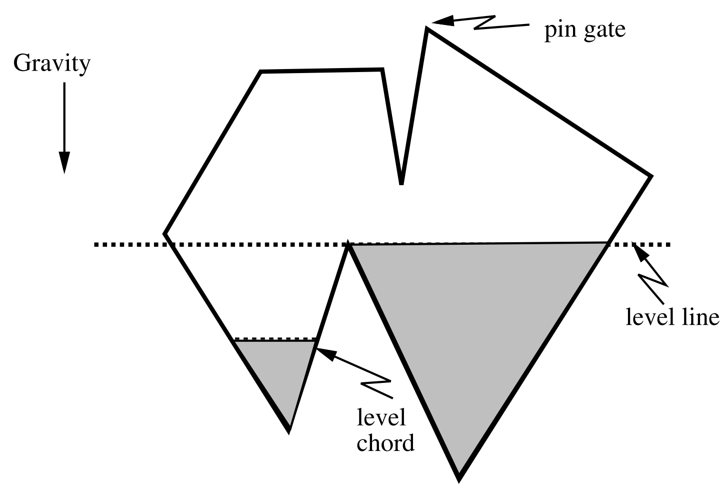


Figure 1: Illustration for definitions.

When the level line coincides with the pin gate, we say the polygon is *maximally injected*. A region containing air in a maximally injected polygon is called an *air pocket*. A polygon is said to be *1-fillable* if there exists a pin gate and direction of gravity such that when the polygon is injected through the pin gate, there are no air pockets when the polygon is maximally injected. If the polygon is not filled, the region inside the polygon and above the level line contains air. Similarly, the sub-polygon containing the level chord, below the level line inside the polygon, contains air above the level chord. We call the highest point (there may be more than one) of an air pocket in a maximally injected mold, the *peak* of the air pocket.

The notion of fillability can be extended in the following two ways. A simple polygon P said to be *k-fillable* provided there exists an orientation where k venting holes are needed to completely fill P with the given orientation.

A simple polygon is *k-fillable* with re-orientation provided that the polygon can be re-oriented and filled from a new pin-gate after partial filling from an initial direction and pin-gate. The k refers to the number of times that the polygon needs to be re-oriented before it is completely filled. We assume that after each completion of a partial filling, the liquid that is injected into the polygon hardens and remains attached to the mold (polygon) boundary. Notice that both definitions are equivalent when $k = 1$. Unless stated otherwise, we will always refer to *k-fillable* as filling from a fixed orientation.

2.2 The Decision Problem

The first problem we address is the decision problem, to determine if a given object, modelled as an n-sided simple polygon, in a fixed orientation with a point representing the pin gate can be filled without any air pockets forming.

We first define the notion of a monotonic chain since it plays an important role in determining the orientation sought for. A polygonal chain P is *monotonic* with respect to direction θ if the projections of the vertices $p_1, p_2, ..., p_n$ of P onto a line L orthogonal to θ are ordered on L as the vertices in P.

Theorem 2.1 (Bose and Toussaint[BT93]) A polygon P is 1-fillable if and only if $\forall\, p \in P$, there exists a path from p to the pin gate that is monotonic with respect to the direction of gravity.

Given this characterization, a simple linear time algorithm can solve the problem. In fact, even if the polygon has holes, the characterization and the complexity of the decision problem remain unchanged.

Corollary 2.2 (Bose and Toussaint[BT93]) A polygon P with holes is 1-fillable if and only if $\forall\, p \in P$, there exists a path from p to the pin gate that is monotonic with respect to the direction of gravity.

2.3 Determining all Directions of Fillability

Suppose that we are simply given a polygon and asked whether there exists a direction of gravity allowing the polygon to be 1-fillable with respect to that direction. For example, the polygon in Figure 2 is 1-fillable in one orientation but not in another. A direction that allows 1-fillability is called a *feasible* direction of gravity.

Theorem 2.2 (Bose and Toussaint[BT93]; Fekete and Mitchell[FM93]) All of the directions of gravity that allow the 1-fillability of a simple polygon or a polygon with holes can be found in $O(n \log n)$ time.

An $\Omega(n \log n)$ time lower bound exists for this problem, proving the optimality of the above

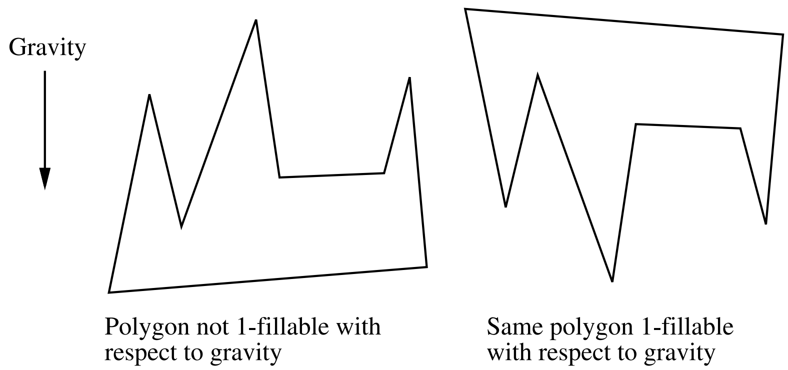


Figure 2

algorithms.

Theorem 2.3 (Fekete and Mitchell[FM93]) Finding all the directions of gravity that allow the 1-fillability of a simple polygon requires $\Omega(n \log n)$ time.

If a polygon P is not 1-fillable from any direction, then injecting P from every direction will result in at least one air pocket. In such a situation, venting holes need to be placed in order to fill the polygon from one orientation without any air pockets. Minimizing the number of venting holes is desirable in this case.

Theorem 2.4 ([Bose and Toussaint[BT93]; Fekete and Mitchell[FM93]) The minimum number of venting holes needed to fill a simple polygon from one fixed direction can be computed in $O(n \log n)$ time.

The lower bound given in Theorem 2.3 applies also to the problem of Theorem 2.4 and thus this algorithm is also optimal.

2.4 Fillability of Certain Classes of Polygons

In this subsection we review the relationships between certain known classes of polygons and fillability. The relationships are important since for certain special classes of polygons, the problem can be solved in linear time as opposed to $\theta(n \log n)$ time required for an arbitrary simple polygon.

A polygon P is *monotone* with respect to direction θ if the boundary of P can be partitioned into two chains that are monotonic with respect to θ. (Refer to Figure 3).

Theorem 2.5 (Bose and Toussaint[BT93]) A monotone polygon is 1-fillable if gravity is oriented in the direction of monotonicity.

A polygon is *edge visible* if there is an edge in the polygon such that for all the points in the polygon there is at least one point on the edge that *sees* it (i.e. the line segment between them does not intersect the exterior of the polygon). A polygon is *open-edge visible* if there is an edge such that all points in the polygon are visible from some point other than the end-points of the edge. (Refer to Figure 4).

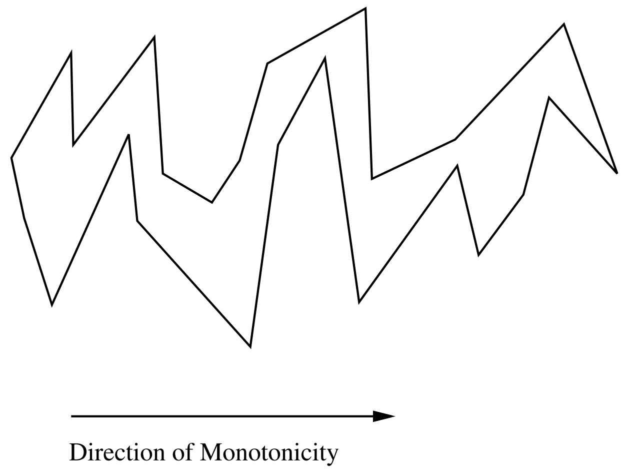


Figure 3: A monotone polygon.

Theorem 2.6 (Bose and Toussaint[BT93]) An open-edge visible polygon P is 1-fillable.

This immediately implies the following.

Corollary 2.2 (Bose and Toussaint[BT93]) Any polygon that is weakly visible from a chord is 2-fillable with re-orientation.

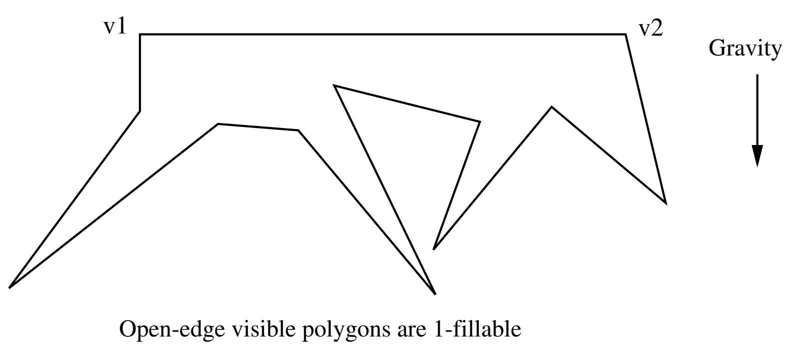
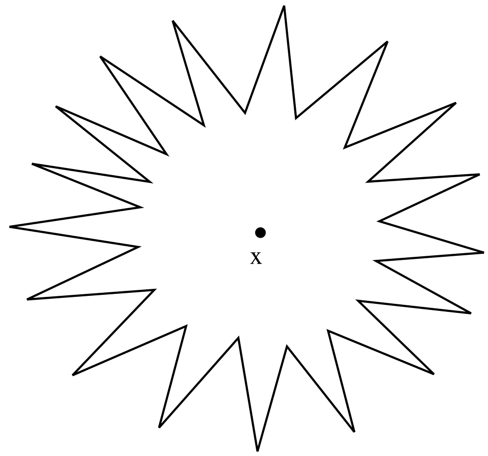


Figure 4: Open edge visible polygon

A *star-shaped polygon* is a polygon that contains at least one point x from which all points of the polygon are visible. A star-shaped polygon may not necessarily be 1-fillable (see Figure 5), but it is weakly visible from any chord containing a point of the kernel. The kernel is the entire collection of points from which the polygon is star-shaped.

Theorem 2.7 (Bose and Toussaint[BT93]) A star-shaped polygon is not necessarily 1-fillable but can always be 2-filled with re-orientation in $O(n)$ time.

A polygon is *clam-shell* if the boundary of the polygon can be partitioned into two chains such that each chain can be removed from the mold by a single translation (not necessarily in a common direction). Clam-shell polygons are studied in Rappaport and Rosenbloom[RR92] where the following result is proved. (Refer to Figure 6).



Star-shaped polygon with x in kernel

Figure 5

Theorem 2.8 (Rappaport and Rosenbloom[RR92]) A polygon is clam-shell if and only if the boundary can be decomposed into two chains, each monotonic to an arbitrary direction. Clam-shells can be recognized in linear time.

Therefore, this class is a generalization of monotonic polygons, but still retains the property of 1-fillability.

Theorem 2.9 (Bose and Toussaint[BT93]) A clam-shell polygon is 1-fillable.

A polygon P is *L-convex* if $\forall\ x,y \in$ P, $\exists\ z \in\ P$ such that $[xz] \in\ P$ and $[yz] \in\ P$. (Refer to Figure 7). From the definition of *L*-convexity it follows that a star-shaped polygon is *L*-convex since all points are seen by a kernel point.

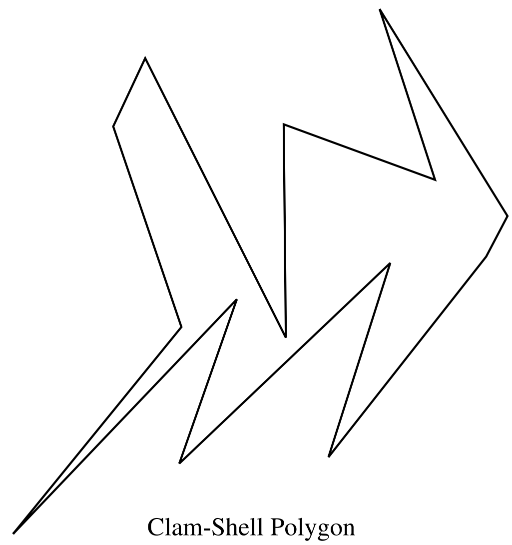
Lemma 2.1 (Horn and Valentine [HV49]) If P is an *L*-convex polygon, it has the property that for every point $x \in\ P$, there exists a chord of the polygon containing x from which P is weakly visible.

The lemma proved in [HV49] is more general and applies to *L*-convex sets, but the proof is non-constructive. An algorithm to find such a chord is given in [BT93], and from Corollary 2.2 together with the example of the star-shaped polygon that is not 1-fillable (Figure 5), we obtain the following:

Theorem 2.10 (Bose and Toussaint[BT93]) An L-convex polygon is not necessarily 1-fillable but always 2-fillable with re-orientation. Furthermore, such an orientation can be found in $O(n \log n)$ time.

The link diameter is a convenient method of classifying simple polygons. For example, an L-convex polygon is a polygon with *link diameter* 2, that is, every pair of points in the polygon can be connected by a polygonal path with at most 2 edges (links) that does not leave the polygon.

Theorem 2.11 (Fekete and Mitchell[FM93]) There exist polygons with link diameter 4 and link diameter 5 that require $\Omega\ (n)$ fillings with fixed directions of gravity and with re-orientation.



Clam-Shell Polygon

Figure 6

A polygon is weakly-externally visible if for every point x on its boundary there exists an infinite ray emanating from that point in some direction that intersects the boundary only at x. In fact, the class of weakly-externally visible polygons is in some sense the largest class of interest since it is the largest class of polygons that allows the pin gate to be placed anywhere on its boundary, if we imagine the pin gate as a line probe from a region outside of the convex hull of the polygon.

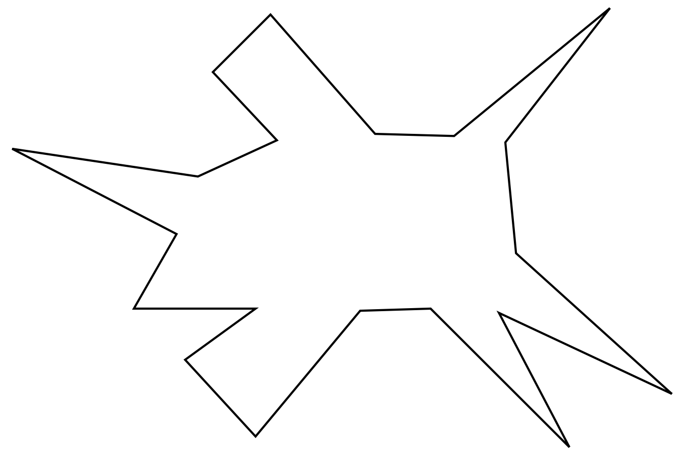
Theorem 2.12 (Fekete and Mitchell[FM93]) A weakly-externally visible polygon is not k-fillable for any constant k. There exist weakly-externally visible polygons that require $\Omega(n)$ fillings with fixed directions of gravity and with re-orientation.

We summarize the relationships between fillability and classes of polygons by the chart in figure Figure 8.

2.5 Alternate Geometric Models

So far, we have assumed that a liquid or molten metal is poured into the mold, and that gravity is the main force acting on the liquid. This assumption although easy to visualize is not useful for gravity casting of two-dimensional objects because 2-dimensional objects are not built this way in practice. Nevertheless, the ideas developed here are relevant to gravity casting of three-dimensional objects discussed in the next section and serve as an introduction to the ideas in 3 dimensions. On the other hand, many thin objects (such as subway seats) are built by injection molding with polymers, where the liquid is injected into the mold under pressure. The main force acting on the liquid is no longer gravity but the pressure used to inject the liquid into the mold. In this case, the liquid fills from the pin gate rather than entering from the pin gate, dripping to the bottom of the object, and filling from there. We assume the pin gate is placed at the lowest point in the object and a venting hole at the highest point in the object. When the liquid is injected under pressure into the polygon, the level of the liquid rises in the direction of the pressure. The lowest horizontal line such that all the liquid in the polygon is contained below it, is defined as the *level line*. This geometric model is called the *pressure molding* model. In this model, we say that an object is k-fillable if k

venting holes are needed in any fixed orientation. While in practice the "front" of the advancing



L-Convex Polygon

Figure 7

liquid (molten plastic, etc.) may not do so in a straight line when a point pin-gate is used, the straight line serves as an approximation. Furthermore, when a linear (rather than point) pin-gate is used the approximation may be quite good. Therefore the results of section 2 are relevant to the injection molding of thin or flat 3-dimensional objects with polymer type substances.

Theorem 2.13 A polygon that is k-fillable in the gravity casting model is also k-fillable in the pressure model.

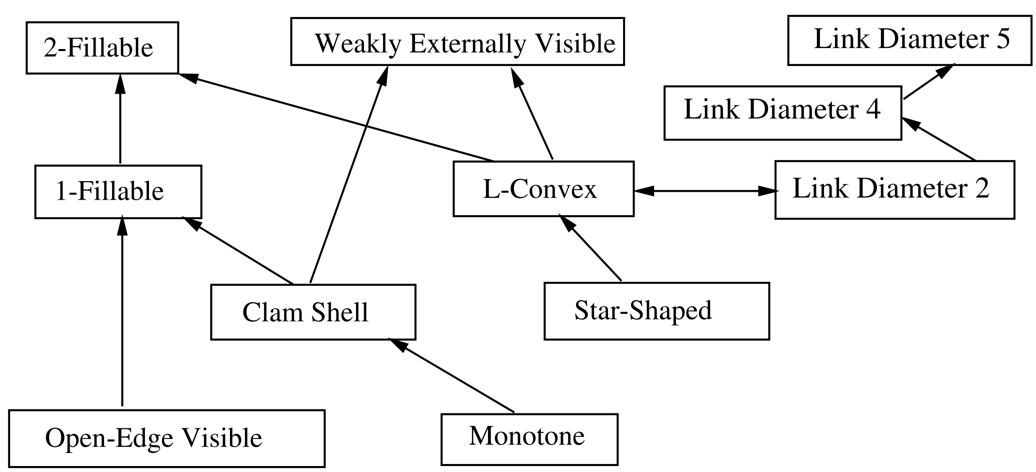


Figure 8

3 Molding in Three Dimensions

In this section, we summarize the three-dimensional aspects of the mold filling problem. These results are relevant to gravity-casting of 3-dimensional objects using molten metal for example.

3.1 Notation and Preliminaries

A *polyhedron* in E^3 is a solid whose surface consists of a number of polygonal faces. A polyhedron is *simple* if its surface can be deformed continuously into the surface of a sphere. A mold will be modelled by a simple polyhedron. As we are dealing only with simple polyhedra, we will refer to them as polyhedra.

A polyhedron partitions the space into two disjoint domains, the *interior* (bounded) and the *exterior* (unbounded). We will denote the open interior of the polyhedron P by $int(P)$, the boundary by $bd(P)$, and the open exterior by $ext(P)$. The boundary is considered part of the polyhedron; that is, $P = int(P) \cup bd(P)$.

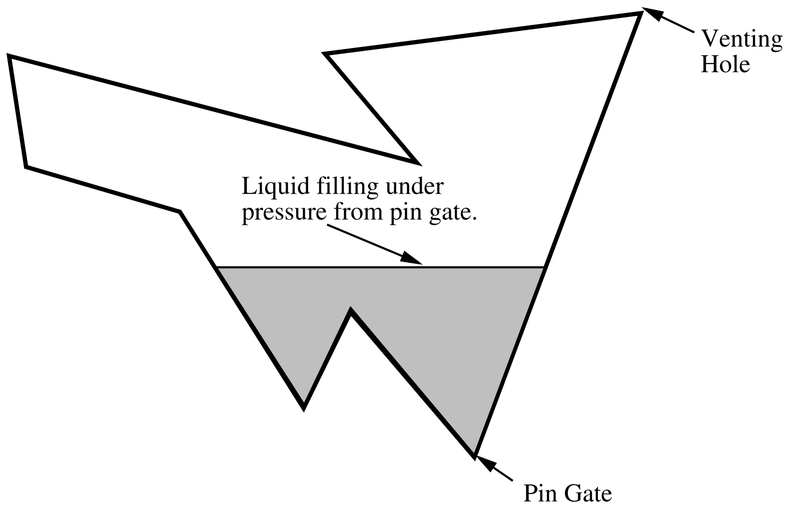


Figure 9

If we intersect a polyhedron with an arbitrary plane, the result is a collection (possibly empty) of simple polygons (or line segments or points) lying on the plane. A polygon in this collection will be referred to as a *sectional* polygon. Notice that a sectional polygon divides the polyhedron into two simple polyhedra. Thus in this sense a sectional polygon is the three dimensional equivalent to a chord in a polygon in 2 dimensions.

We assume the gravity casting model. When a direction of gravity is not specified, we assume, for simplicity of exposition, that gravity points in the negative z-direction. Thus, if only one pin gate is used, we assume it to be a point on the boundary with the highest z-coordinate, since otherwise, the polyhedron cannot be completely filled. Unless stated otherwise, we will refer to *k-fillable* as filling from a fixed orientation.

3.2 The Decision Problem

Theorem 3.1 (Bose, van Kreveld and Toussaint[BVT93]; Fekete and Mitchell[FM93]) A polyhedron P is 1-fillable if and only if the orientation of P has precisely one local maximum in direction $+z$.

To test the 1-fillability of a polyhedron P with respect to gravity g, only the number of local maxima with respect to g need to be determined. Deciding if a vertex is a local maximum can be accomplished in time linear in the degree of the vertex. This immediately gives a linear time algorithm to determine whether or not a polyhedron is 1-fillable from a fixed orientation.

Theorem 3.2 (Bose, van Kreveld and Toussaint[BVT93]; Fekete and Mitchell[FM93]) Given a polyhedron P, in $O(n)$ time it can be determined whether or not the polyhedron is 1-fillable with respect to gravity.

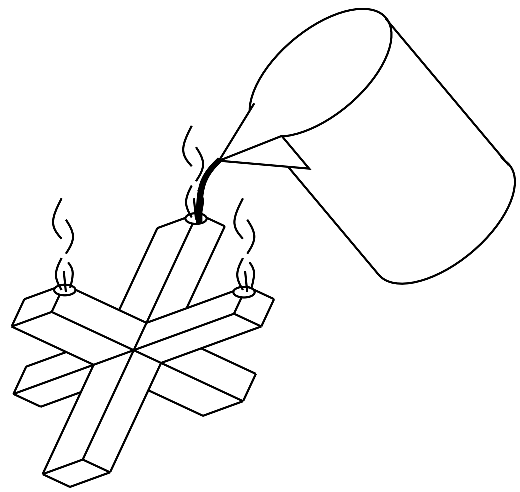


Figure 10: Gravity casting a star-shaped object.

3.3 Determining all Directions of Fillability

Suppose that we are given a polyhedron and asked whether there exists a direction of gravity allowing the polyhedron to be 1-fillable with respect to that direction. An $O(n^2)$ time algorithm exists to find the orientation of a given polyhedron P that minimizes the number of venting holes needed to ensure a complete fill from a fixed orientation. The algorithm has two stages. In the first stage, the fillability problem is transformed to a planar problem for a set of convex (possibly unbounded) polygons that cover the plane.

Lemma 3.1 (Bose, van Kreveld and Toussaint[BVT93]) In $O(n)$ time, one can transform the problem of k-fillability to the problem of finding a point in the plane covered by only k convex polygons.

Each convex polygon represents the set of directions that make the associated vertex a local maximum. Thus, the next step in the algorithm involves solving the following problem: 'Given a set Q of n convex, but not necessarily bounded, polygons in the plane, with total complexity $O(n)$, find a point that is covered by the minimum number of polygons of Q.' The cell in the subdivision induced by Q covered by the minimum number of polygons represents the set of directions that has the minimum number of local maxima. The time to find this cell is bounded by the total number of cells which is $O(n^2)$.

Theorem 3.3 (Bose, van Kreveld and Toussaint[BVT93]; Fekete and Mitchell[FM93]) Given a simple bounded polyhedron P in 3-space, one can find in $O(n^2)$ time an orientation for P such that P is fillable with the minimum number of venting holes.

3.4 A Reduction from Covering to 1-Fillability

A *pseudo-lower bound* on the complexity of this problem is given by a reduction from the problem "A+B=C" to mold filling. The problem "A+B=C" is defined as follows: Given three sets A, B and C of n real numbers each, decide if there exists $a \in A$, $b \in B$ and $c \in C$ such that $a+b=c$. The best known algorithm for "A+B=C" uses $O(n^2)$ time. Several other geometric problems also reduce to this problem, such as: 'Given a set of n points in the plane, are there three collinear points?' and 'Given a set of n rectangles in the plane, do they cover a given rectangle "RECT" completely?'. In fact, the rectangle covering problem can be reduced to the filling problem [BVT93], and "A+B=C" can be reduced to the rectangle covering problem [GO93]. Thus the quadratic bound seems difficult to improve even to determine whether there is an orientation with only one local maximum.

Theorem 3.4 (Bose, van Kreveld and Toussaint[BVT93])The rectangle covering problem can be reduced to the 1-fillability problem in $O(n \log n)$ time.

3.5 Fillability of Certain Classes of Polyhedra

In this subsection, we outline the relationships between the notion of fillability and certain known classes of restricted polyhedra. These results are relevant to the manufacturing industry because in practice many objects are not modelled by polyhedra of arbitrary shape complexity.

A polyhedron P is *weakly monotonic* in direction l if there exists a direction l such that the intersection, of each plane orthogonal to l that intersects P, is a *simple polygon* (or a line segment or point). The direction l is referred to as the direction of monotonicity.

Note that there exist many different classes of simple polygons [Or87], [PS85], [To90], [Tou]. By substituting one of these classes for the word *simple* in the above definition, a score of families of *weakly monotonic* polyhedra are obtained. Thus we say that if all the intersections are *convex* polygons, we have a weakly monotonic polyhedron in the *convex sense*. If the intersections are *monotone* polygons, then we have a weakly monotonic polyhedron in the *monotone sense*, and so on. Refer to Figure 11.

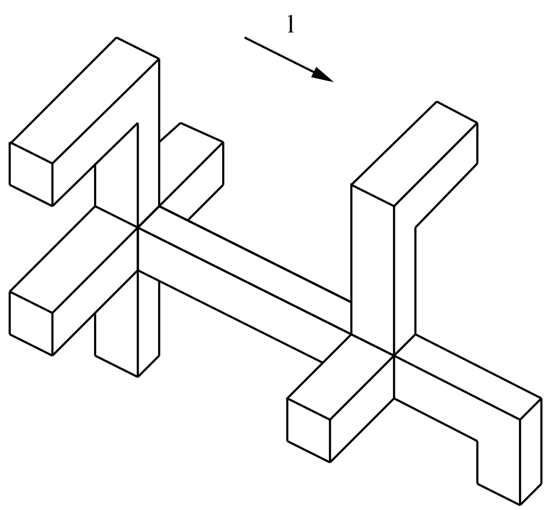


Figure 11: Weakly Monotonic Polyhedron

Theorem 3.5 (Bose, van Kreveld and Toussaint[BVT93]) A weakly monotonic polyhedron P is 1-fillable if it is oriented such that gravity points in the direction of monotonicity.

A polyhedron P is *facet visible* if there is a facet of the polyhedron from which all the points in the polyhedron are weakly visible. A polyhedron P is *open-facet visible* if there is a facet f in P such that all points p are visible from some point x on f that is not on the boundary of the facet.

Theorem 3.6 (Bose, van Kreveld and Toussaint[BVT93]) An open-facet visible polyhedron P is 1-fillable.

Theorem 3.7 (Bose, van Kreveld and Toussaint[BVT93]) Every polyhedron that is weakly visible from a sectional polygon is 2-fillable with re-orientation.

A *star-shaped polyhedron* is a polyhedron that contains at least one point x from which all points of the polyhedron are visible (see Figures 10 and 12 for examples of star-shaped polyhedra). Such a polyhedron may not necessarily be 1-fillable (see Figure 12). In fact, if a star-shaped polyhedron is filled from one fixed orientation, it may need $\Omega(n)$ venting holes.

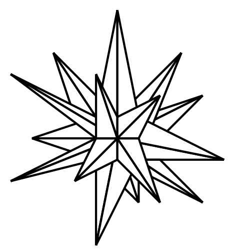


Figure 12: Star-shaped polyhedron.

Theorem 3.8 (Bose, van Kreveld and Toussaint[BVT93]) A star-shaped polyhedron is not necessarily 1-fillable but can always be 2-filled with re-orientation in $O(n)$ time.

We can classify polyhedra by their link diameter. A polyhedron P has *link diameter k* if every pair of points in P can be connected by a polygonal path with at most k edges that does not leave the polyhedron.

Theorem 3.9 (Fekete and Mitchell[FM93]) A polyhedron with link diameter 1 is 1-fillable.

Theorem 3.10 (Fekete and Mitchell[FM93]) A polyhedron with link diameter 2 is 2-fillable with re-orientation.

Theorem 3.11 (Fekete and Mitchell[FM93]) There exist polyhedra with link diameter 3 that require Ω (n) fillings, even if they are weakly externally visible.

4 Casting

Casting consists of filling the region bounded by two or more cast parts with a material such as a molten metal, after which the cast parts are removed. The removal of the cast parts without breaking them imposes certain restrictions on the shape of the object to be constructed. For *sand casting* (see e.g. [El88, WO81]), only two cast parts are used. To construct the cast parts, a prototype of the object is first obtained (See Figure 13).

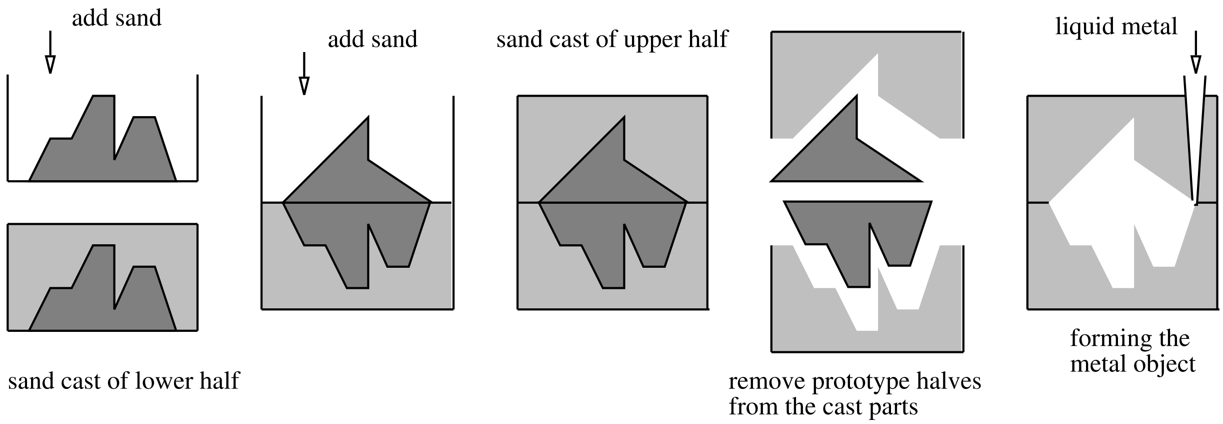


Figure 13: Construction of an object by sand casting, using two halves of the object as prototypes.

The prototype is then divided into two parts along a plane. The face along which the part is cut, is referred to as the base. The first cast part is made by placing the base of the part on a flat surface, and then adding sand around it. The part is then rotated such that the base is facing up. The other part is placed such that the bases coincide and the second cast part is built by adding sand around this part. An opening into the cavity is maintained during the construction of the second cast part. This completes the construction of the cast of the prototype object. To build a metal rendition of the prototype object with this cast, liquid metal is poured into the opening until it fills the cavity. After the metal solidifies, the cast parts are removed from the object. To be able to re-use the cast parts when manufacturing many objects in this way, it is necessary that the two cast parts be removed without breaking the cast. Also, for manufacturing methods related to sand casting, such as other metal casting methods [El88, WO81] and injection molding and blow molding of plastics [Is87, Whe87], it must be possible to remove the constructed object from the cast parts without breaking the cast. Thus for many different manufacturing methods involving casting, the geometry of the object determines its feasibility of construction.

We note that more complicated objects can be made by using cores and inserts [El88,Is87,WO81,Whe87]. However, their use slows down the manufacturing process and makes it more costly. Thus to be cost efficient, cores and inserts should be avoided.

4.1 Casting Two-dimensional Objects

In this subsection, we review the castability of polygons. A polygon is considered to be castable if the boundary can be decomposed into two chains such that each chain can be translated away from the polygon without intersecting the interior. Rosenbloom and Rappaport [RR92] proved the following.

Theorem 4.1 (Rosenbloom and Rappaport [RR92]) A simple polygon is castable if and only if its boundary can be decomposed into two monotone chains.

This characterization gives rise to a simple efficient algorithm to decide whether a simple polygon is castable.

Theorem 4.2 (Rosenbloom and Rappaport [RR92]) In O(n) time one can decide if a simple polygon is castable.

4.2 Casting Three-dimensional Objects

This sub-section addresses casting of polyhedral objects [BBV93]. For a polyhedron P, let ∂P denote the boundary of P, and for a plane h, let h^+ and h^- refer to the open half spaces above and below h. In geometric terms, castability can be defined as follows:

Definition 4.1 A simple polyhedron P is *castable* provided there exists a plane h such that $h^+ \cap \partial P$ is a *weak* terrain in some orientation, and $h^- \cap \partial P$ is a *weak* terrain in some orientation. The plane h is called the *casting plane*. (A definition of a terrain can be found in sub-section 5.5. A *weak* terrain is a terrain that may contain vertical edges and facets.)

To manufacture a polyhedron P that is castable, first determine a casting plane h for P. Then the cast parts C_1 and C_2 are made from the prototype halves $h^+ \cap \partial P$ and $h^- \cap \partial P$. Since P is castable, the prototype halves can be removed from the cast parts, and later the manufactured object can be removed from the cast parts. Although the algorithms in [BBV93]deal with the sand casting process, they can be applied in related processes as well. In [BBV93] three versions of the castability problem are considered. They differ in the way the cast may be removed from the polyhedron P.

1. The two cast parts must be removed from P by one translation each, in opposite directions, and normal to the casting plane (orthogonal cast removal).
2. The two cast parts must be removed from P by one translation each, and in opposite directions (opposite cast removal).
3. The two cast parts must be removed from P by one translation each, in arbitrary directions (arbitrary cast removal).

In manufacturing, developing machines that perform orthogonal and opposite case removal is much simpler than machines that perform arbitrary cast removal. In fact, opposite cast removal seems to be the most popular technique used ([Pri87], [PC93]). Furthermore, if orthogonal or opposite cast removal is possible, it can be determined more efficiently. We summarize the complexity of the different algorithms [BBV93] developed for the casting problem.

	orthogonal	opposite	arbitrary
convex polyhedra	$O(n \log^2 n)$	$O(n \log^2 n)$	$O(n^2 \log n)$
simple polyhedra	$O(n^2)$	$O(n^2)$	$O(n^2 \log n)$

5 Layer Deposition Methods

In this section, we consider the problem of deciding whether or not a design is feasible for a CAD/CAM system developed and patented by 3D Systems of Sylmar, CA that employs a process called *stereolithography*. For details the reader is referred to [ABB93]. The components of the stereolithography manufacturing process consist of a vat of liquid photocurable plastic, a computer controlled table T on a stand S that can be moved up and down in the vat and a laser L above the vat that can shine on the surface of the liquid plastic and can move in a horizontal plane. The system works as follows. At the first step the table is just below the surface of the plastic and the laser is controlled to move about so that the light shines on the surface of the plastic and *draws* the bottom-most cross-section of the object A being built. When the laser light contacts the plastic, the plastic solidifies and so the first cross-section of the object is formed and rests on the table. At the next step the table is lowered a small amount to allow liquid to cover the hardened layer and the laser then draws the next cross-section of the object. The light from the laser penetrates the liquid just deep enough so that this cross-section is welded to the lower cross-section produced at the previous step. This process is repeated until the entire object is formed. The direction given by a normal to the table pointing to the laser is called the *direction of formation* for the object. There are some objects that can be formed only if the direction of formation is chosen correctly. Naturally, there are some objects that can not be formed using stereolithography regardless of the direction of formation chosen, such as a sphere.

5.1 Feasibility of Polygonal Objects

In this sub-section we consider the two-dimensional problem where the object A we wish to form is a polygon. Let $e_1, e_2, ..., e_n$ be a clockwise ordering of the edges around A and for $1 \leq i \leq n$, let θ_i be the angle formed by e_i and e_{i+1} in the interior of A (where $e_{n+1} = e_1$). Note that θ_0 will mean θ_n. If edge e_i is such that θ_{i-1} and θ_i are both less than or equal to $\pi / 2$ then e_i is called an *acute edge*. If e_i is an acute edge and at least one of θ_{i-1} or θ_i is strictly less than $\pi /2$ then e_i is said to be a *strictly acute edge*.

Lemma 5.1 ([ABB93]) Given a convex polygon A, the edge e_i is a valid base if and only if e_i is acute.

From convexity it follows that a simple examination of the angles between the edges of a convex object is sufficient to find a valid base if one exists or report that the object is not feasible. For a non-convex object however such local tests on the angles will be insufficient to determine the feasibility of an object. However the angles can be used to narrow the search for a valid base.

If object A placed on horizontally on edge e_i contains a horizontal edge whose points have y-coordinates at least as great as any point in A and such that the x-coordinates of all points in A lie between the x-coordinates of the end-points of the edge, then we label the edge $e_{top}(i)$.

Lemma 5.2 ([ABB93]) Polygon A is feasible if and only if A contains an acute edge e_i and at least one of e_i, e_{i-1}, e_{i+1} or $e_{top}(i)$ (if it exists and $\theta_{i-1} = \theta_i = \pi /2$) is a valid base.

Determining whether or not a polygon has an acute edge can be achieved in $O(n)$ time, where n is the number of vertices in the polygon. Therefore, in $O(n)$ time, the number of possible valid bases can be reduced to 3 or 4.

Lemma 5.3 ([ABB93]) Deciding if an edge e_i of a polygon A is a valid base can be done in $O(n)$ time.

Therefore, we can test a polygon A for feasibility and find a valid base in $O(n)$ time where n is the number of vertices of A.

Theorem 5.1 ([ABB93]) In $O(n)$ time the feasibility of a polygonal object with n vertices can be determined and all valid bases identified when the object is feasible.

5.2 Polyhedral objects

In this subsection we consider the three-dimensional case where the object is a simple polyhedron. We want to find a face f of polyhedron A that is a valid base or determine that A is not feasible.

The following notation will be used in this sub-section. Let A be a polyhedron with n vertices. For face f of A, let $f(1), f(2),..., f(k_f)$ be the faces of A that share at least one edge with f. Let $\theta_i(f)$ be the angle interior to A between the plane P containing f and the plane $P(i)$ containing $f(i)$ about the line of intersection of P and $P(i)$. If $\theta_i(f) \le \pi/2$ for all i, $1 \le i \le k_f$, then f is called an *acute face*. If f is acute and for some i, $\theta_i(f) < \pi/2$, then f is said to be a *strictly acute face*. We mention several properties analogous to those in the previous subsection that give rise to a linear-time feasibility testing algorithm.

In the special case of convex polyhedra, a simple local test on each face suffices to determine feasibility.

Lemma 5.4 ([ABB93]) Let A be a convex polyhedron. Face f is a valid base if and only if f is acute.

We now turn our attention to polyhedral objects that are not necessarily convex. If the polyhedral object A has a strictly acute face f, then f or one of its adjacent faces must be a valid base if the object is feasible. If f is not strictly acute, we define f_{top} analogously to $e_{top}(i)$ in the previous subsection. We then have the following lemma.

Lemma 5.5 ([ABB93]) If polyhedron A is feasible and f is an acute, but not strictly acute, face then $f, f(1),..., f(k_f)$ or f_{top} is a valid base.

These results were sufficient in the two-dimensional case to reduce the number of candidate bases to at most 4. Unfortunately, in the 3-dimensional case, an acute face f may have $O(n)$ adjacent faces. Nevertheless, we have the following theorem.

Theorem 5.2 ([ABB93]) Polyhedron A is feasible if and only if A contains an acute face f and at least one of the following is a valid base of A: f, f_{top} (if it exists) and at most 4 faces adjacent to f.

Thus, in $O(n)$ time, where n is the number of vertices in polyhedron A, the number of possible valid bases can be reduced to at most 6. Therefore, we can test a polyhedron A for feasibility and find a valid base in $O(n)$ time where n is the number of vertices of A.

Theorem 5.3 ([ABB93]) In $O(n)$ time the feasibility of a polyhedral object with n vertices can be determined and a valid base identified when the object is feasible.

5.3 Variable-Angle Stereolithography

In this sub-section, we characterize the objects, both polygonal and polyhedral, that can be formed with variable-angle stereolithography. Intuitively, in variable-angle stereolithography, one can imagine that as each layer is glued on by the laser, the topmost layer cannot hang over the previous layer by more than the freedom allotted by some constant angle ω. More formally, we say that a point p with y-coordinate y_0 is supported with respect to the direction of formation if there exists a point q with positive y coordinate less than y_0 such that the line segment qp is contained in the object and the smaller angle between the direction of formation and the vector $\overline{qp}$ is less than or equal to ω. Clearly, ω must be less than $\pi / 2$. Notice that variable-angle stereolithography is a generalization of stereolithography. The two are equivalent when ω is zero.

5.3.1 Polygonal Objects

Although the parameter ω enlarges the class of objects that can be formed, the following two key theorems allow the efficient recognition of the polygons that can be formed using variable-angle stereolithography.

Theorem 5.4 ([ABB93]) For fixed ω, a convex polygon has a constant number of edges that can be valid bases. Furthermore, these edges can be obtained in $O(n)$ time.

Theorem 5.5 ([ABB93]) If a polygon A is feasible, then every valid base of A is also a valid base of the convex hull of A.

We remark here that the convex hull of a simple n-vertex polygon can be computed in $O(n)$ time ([MA79], [Mel87]). Therefore we have the following theorem.

Theorem 5.6 ([ABB93]) The feasibility of a simple polygon in variable-angle stereolithography can be determined in $O(n)$ time.

5.3.2 Polyhedral Objects

With respect to polyhedral objects, the parameter ω not only enlarges the class of objects that can be formed but also renders the determination of feasibility more difficult than the two-dimensional case. The following two key theorems are analogous to the two-dimensional case.

Theorem 5.7 ([ABB93]) For fixed ω, a convex polyhedron has a constant number of faces that can be valid bases. These faces can be obtained in $O(n)$ time.

Theorem 5.8 ([ABB93]) If a polyhedron A is feasible, then every valid base of A is also a valid base of the convex hull of A.

The complexity of computing the convex hull of a polyhedron is $O(n \log n)$ ([PS85]) and therefore the algorithm derived by computing the convex hull is sub-optimal. However, in [ABB93] it is shown how to circumvent the computation of the convex hull and obtain the following result.

Theorem 5.9 ([ABB93]) The feasibility of a polyhedron can be determined in time $O(n / \tau^2)$, where $\tau = \pi / 2 - \omega$.

5.4 Relation to Injection Molding

In [BVT93], it is shown that given an arbitrary polyhedron on n vertices, in $O(n^2)$ time all the orientations that allow the polyhedron to be filled from one pin gate can be found. It is also shown that this problem is n^2-hard [GO93], which leads one to believe that improving this time complexity will be difficult. However, for the restricted case of a polyhedron that can be built using variable-angle stereolithography, the following theorem relates the two manufacturing processes.

Theorem 5.10 ([ABB93]) A polyhedral object formed by variable-angle stereolithography can be formed by injection molding using only one pin gate.

5.5 Relation to Polyhedral Terrains

Polyhedral terrains are important structures in geographic information systems ([Lee91], [GL89]) and computational geometry ([CS89], [DFP86], [BST92]). A polyhedral terrain is defined as follows.

A terrain T is a two-dimensional triangulated polyhedral surface with n vertices $V = \{v_1, v_2,..., v_n\}$. Each vertex v_i is specified by three real numbers (x_i, y_i, z_i) which are its cartesian coordinates and z_i is referred to as the height of vertex v_i. It is convenient to assume that z_i is non-negative so that if the X-Y plane is associated with *sea-level*, no points on the terrain are below sea-level. Let $P = \{p_1, p_2,..., p_n\}$ denote the orthogonal projections of the points $V = \{v_1, v_2,..., v_n\}$ on the X-Y plane, i.e., each point p_i is specified by the two real numbers (x_i, y_i). It is assumed that the set $P = \{p_1, p_2,..., p_n\}$ is in general position, i.e., no three points are collinear and no four are cocircular so that the projections of the edges of the polyhedral surface onto the X-Y plane determine a triangulation of P (hence the term triangulated polyhedral surface). We refer to the triangulation as the underlying triangulated planar graph associated with the terrain. Therefore we can view a terrain T as the graph of a polyhedral function $z=F(x,y)$, defined over the convex hull of P. Sometimes a polyhedral terrain is assumed to be a monotone polyhedral surface, i.e., a polyhedral surface having exactly one intersection with every vertical line [CS89]. In our case we assume the stronger condition that the intersection of every vertical line in the interior of convex hull of P with the polyhedral surface is a single point. Intuitively, a monotone terrain admits vertical walls whereas our definition does not. Since the orthogonal projection of T onto the X-Y plane is a planar straight-line subdivision or map, it follows that T has $O(n)$ edges and $O(n)$ triangular faces. A polyhedron P is a solid terrain provided that it can be positioned such that it is vertically visible from one of its faces.

We have the following theorem.

Theorem 5.11 ([ABB93]) Determining if a polyhedron P is a solid terrain can be achieved in $O(n)$ time.

In fact, the polyhedra that can be formed by variable-angle stereolithography provide a generalization of the standard polyhedral terrains studied in geographic information systems. The algorithms of [ABB93] recognize whether a polyhedral surface is a terrain that allows overhangs, a more realistic terrain.

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