COMPUTATIONAL GEOMETRY FOR CAD/CAM

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ABSTRACT

The two fundamental questions that arise in CAD/CAM systems concerning every type of manufacturing process that is intended to realize a design are: (1) Given a designed object, can it be constructed using a particular process? (2) Given that a designed object can be built using a particular process, what is the *best* way to construct the object? A brief survey is presented of computational geometric tools for designing algorithms to answer these types of questions, as well as key results already obtained, for several manufacturing processes such as gravity casting, injection molding, and stereolithography.

1 Introduction

In the manufacturing industry, there are many different processes such as molding, casting, NC machining, laser sculpting, automated welding and 3-D printing (stereolithography), available to construct an object [Al88], [Mu90], [As91]. In fact, these processes are also finding application to fine arts such as sculpting [Ha89]. However, every manufacturing process, regardless of the intended application, imposes certain restrictions on the types of objects that can be constructed as well as the way a given object may be built. For example, a sphere cannot be built using stereolithography, but can be easily built using injection molding. Also, a cube can be more easily constructed using stereolithography when placed on one of its faces. Two fundamental questions arise concerning every type of manufacturing process:

1. Given an object design, can the object be constructed using a particular process?
2. Given that an object can be built using a particular process, what is the *best* way to construct the object?

The latter question gives rise to many different problems depending on how *best* is qualified. The geometry of the object, coupled with the restrictions imposed by the particular manufacturing process under consideration, play a vital role in determining the answer to these questions.

The importance of these questions is quite evident. For example, when designing an object to be built by a certain type of manufacturing process, currently an engineer must always keep in mind the process that is to be used to construct the object. This limits the creativity of the engineer since the question of design feasibility must always be kept in mind while creating the object. In fact, the engineer is never really quite sure whether the object can be built since no formal methods exist to determine the feasibility of an object for most manufacturing processes. To resolve this problem, practical algorithms are needed to determine, given an arbitrary object, whether or not it can be built using any of the known manufacturing processes. The benefits of such a CAD/CAM system would be two-fold. Firstly, an engineer would have an algorithm to verify whether an object can be created using the manufacturing process intended (i.e. a type of automatic design verification). Secondly, a list of the possible manufacturing processes that can build a particular object would allow an engineer to design something and then determine which manufacturing process would be most cost efficient.

In the last two years, there has been a flurry of activity concerning the study of the geometric and computational aspects of manufacturing processes from this perspective. In this paper, we outline such recent developments for injection molding, gravity casting, and stereolithography.

2 Molding in Two Dimensions

> *"During injection, the mold is tilted into a favorable position that will eliminate surface defects such as bubbles and insure a complete fill. Mold orientation during fill is a cut-and-try process to find the most favorable position."*
> [Pri87]

The main motive for focusing on the geometry of the object to be molded, has been to remove the 'cut-and-try' from the process of finding a favorable mold position. A mold, as defined in [Bo79], refers to the whole assembly of parts that make up a cavity into which liquid is poured to give the shape of the required component when the liquid hardens. Given a mold, establishing whether there exists an orientation that allows the filling of the mold using only one pin gate (the pin gate is the point from which the liquid is injected into the mold) as well as determining an orientation that allows the most complete fill are two major problems in the field of injection molding and gravity casting.

These problems are difficult when the focus is on the fluid dynamics and physics of the whole molding process. To date, only heuristics have been proposed as solutions to these two general problems because of the complexity involved [Pri87], [Mac89], [Bo79], [HT92]. However, when viewed from a purely geometric perspective, optimal solutions do exist. In this section, we review the techniques developed in the two-dimensional case (where the mold is modelled as a polygon) which find applications to polymer molds. In practice many 3-dimensional objects are almost flat so that in effect they can be considered as 2-dimensional. Therefore the 2-dimensional theory is more important than may appear at first glance.

2.1 Preliminaries

We assume that the liquid (or molten metal) being poured into the mold has small viscosity and small surface tension much like water. The point on the polygon boundary from which the liquid is poured into the polygon is called the *pin gate*. We assume that the pin gate is the only point from which air is allowed to escape unless stated otherwise. A *venting hole* is a point from which only air and no liquid is allowed to escape. We assume that neither the liquid being poured into the mold, nor the air in the mold are compressible. Finally, we assume that air cannot bubble out through the liquid. This geometric model is called the *gravity casting* model.

When liquid is poured into a polygon, the level of the liquid rises in the direction opposite that of gravity. The lowest horizontal line such that all the liquid in the polygon is contained below it, is defined as the *level line*. It is possible for the level line to be higher than the level of the liquid in some section of the polygonal mold. Therefore we define a *level chord* to be the horizontal chord representing the level of liquid in the sub-polygon lying below the chord.

When the level line coincides with the pin gate, we say the polygon is *maximally injected*. A region containing air in a maximally injected polygon is called an *air pocket*. A polygon is said to be *1-fillable* if there exists a pin gate and direction of gravity such that when the polygon is injected through the pin gate, there are no air pockets when the polygon is maximally injected. If the polygon is not filled, the region inside the polygon and above the level line contains air. Similarly, the sub-polygon containing the level chord, below the level line inside the polygon, contains air above the level chord. We call the highest point (there may be more than one) of an air pocket in a maximally injected mold, the *peak* of the air pocket.

The notion of fillability can be extended in the following two ways. A simple polygon P said to be *k-fillable* provided there exists an orientation where k venting holes are needed to completely fill P with the given orientation.

A simple polygon is *k-fillable* with re-orientation provided that the polygon can be re-oriented and filled from a new pin-gate after partial filling from an initial direction and pin-gate. The k refers to the number of times that the polygon needs to be re-oriented before it is completely filled. We assume that after each completion of a partial filling, the liquid that is injected into the polygon hardens and remains attached to the mold (polygon) boundary. Notice that both definitions are equivalent when $k = 1$. Unless stated otherwise, we will always refer to *k-fillable* as filling from a fixed orientation.

2.2 The Decision Problem

The decision problem is to determine if a given object, modelled as an n-sided simple polygon, in a fixed orientation with a point representing the pin gate can be filled without any air pockets forming. We first define the notion of a monotonic chain since it plays an important role in determining the orientation sought for. A polygonal chain P is *monotonic* with respect to direction θ if the projections of the vertices of P onto a line L orthogonal to θ are ordered on L as the vertices in P.

Theorem 2.1 (Bose and Toussaint [BT93]) A polygon P is 1-fillable if and only if for every point p in P, there exists a path from p to the pin gate that is monotonic with respect to the direction of gravity.

From this characterization a simple linear time algorithm can be obtained to solve the problem. In fact, even if the polygon has holes, the characterization and the complexity of the decision problem remain unchanged.

2.3 Determining all Directions of Fillability

Suppose that we are simply given a polygon and asked whether there exists a direction of gravity allowing the polygon to be 1-fillable with respect to that direction. A direction that allows 1-fillability is called a *feasible* direction of gravity.

Theorem 2.2 (Bose and Toussaint [BT93]; Fekete and Mitchell [FM93]) All the directions of gravity that allow 1-fillability of a simple polygon or a polygon with holes can be found in $O(n \log n)$ time.

Theorem 2.3 (Fekete and Mitchell [FM93]) Finding all the directions of gravity that allow 1-fillability of a simple polygon requires $\Omega(n \log n)$ time.

If a polygon P is not 1-fillable from any direction, then injecting P from every direction will result in at least one air pocket. In such a situation, venting holes need to be placed in order to fill the polygon from one orientation without forming any air pockets. Minimizing the number of venting holes is desirable in this case because constructing venting holes is costly and yields imperfections in the final object manufactured.

Theorem 2.4 ([Bose and Toussaint [BT93]; Fekete and Mitchell [FM93]) The minimum number of venting holes needed to fill a simple polygon from one fixed direction can be computed in $O(n \log n)$ time.

The lower bound given in Theorem 2.3 applies also to the problem of Theorem 2.4 and therefore this algorithm is also optimal.

2.4 Fillability of Certain Classes of Polygons

The relationships between certain known classes of polygons and fillability are important since for certain special classes of polygons, the problem can be solved in linear time in contrast with to $\theta(n \log n)$ time required for an arbitrary simple polygon.

A polygon P is *monotone* with respect to direction θ if the boundary of P can be partitioned into two chains that are monotonic with respect to θ.

Theorem 2.5 (Bose and Toussaint [BT93]) A monotone polygon is 1-fillable if gravity is oriented in the direction of monotonicity.

A polygon is *edge visible* if there is an edge of the polygon such that for every point in the polygon there is at least one point on the edge that *sees* it (i.e. the line segment between them does not intersect the exterior of the polygon). A polygon is *open-edge visible* if there is an edge such that every point in the polygon is visible from some point other than the end-points of the edge in question.

Theorem 2.6 (Bose and Toussaint [BT93]) An open-edge visible polygon P is 1-fillable.

A *star-shaped polygon* is a polygon that contains at least one point x from which all points of the polygon are visible. A star-shaped polygon may not necessarily be 1-fillable but it is weakly visible from any chord containing a point of the kernel. The kernel is the entire collection of points from which the polygon is star-shaped.

Theorem 2.7 (Bose and Toussaint [BT93]) A star-shaped polygon is not necessarily 1-fillable but can always be 2-filled with re-orientation in $O(n)$ time.

A polygon is *clam-shell* if the boundary of the polygon can be partitioned into two chains such that each chain can be removed from the mold by a single translation (not necessarily in a common direction). Clam-shell polygons are studied in Rappaport and Rosenbloom [RR92] where the following result is proved.

Theorem 2.8 (Rappaport and Rosenbloom [RR92]) A polygon is clam-shell if and only if the boundary can be decomposed into two chains, each monotonic to an arbitrary direction. Clam-shells can be recognized in linear time.

Although this class is a generalization of monotonic polygons, it still retains the property of 1-fillability.

Theorem 2.9 (Bose and Toussaint [BT93]) A clam-shell polygon is 1-fillable.

A polygon P is *L-convex* if $\forall\, x,y \in$ P, $\exists\, z \in P$ such that edge $[xz] \in P$ and edge $[yz] \in P$. From the definition of L-convexity it follows that a star-shaped polygon is L-convex since all points are seen by a kernel point.

Lemma 2.1 (Horn and Valentine [HV49]) If P is an L-convex polygon, it has the property that for every point $x \in P$, there exists a chord of the polygon containing x from which P is weakly visible.

The lemma proved in [HV49] is more general and applies to L-convex sets, but the proof is non-constructive. However, an algorithm to find such a chord is given in [BT93].

Theorem 2.10 (Bose and Toussaint [BT93]) An L-convex polygon is not necessarily 1-fillable but always 2-fillable with re-orientation. Furthermore, such an orientation can be found in $O(n \log n)$ time.

The link diameter is a convenient method of classifying simple polygons. For example, an L-convex polygon is a polygon with *link diameter* 2, that is, every pair of points in the polygon can be connected by a polygonal path with at most 2 edges (links) that does not leave the polygon.

Theorem 2.11 (Fekete and Mitchell [FM93]) There exist polygons with link diameter 4 and link diameter 5 that require $\Omega(n)$ fillings with fixed directions of gravity and with re-orientation.

A polygon is weakly-externally visible if for every point x on its boundary there exists an infinite ray emanating from that point in some direction that intersects the boundary only at x. In fact, the class of weakly-externally visible polygons is in some sense the largest class of interest since it is the largest class of polygons that allows the pin gate to be placed anywhere on its boundary, if we imagine the pin gate as a line probe from a region outside of the convex hull of the polygon.

Theorem 2.12 (Fekete and Mitchell [FM93]) A weakly-externally visible polygon is not k-fillable for any constant k. There exist weakly-externally visible polygons that require $\Omega(n)$ fillings with fixed directions of gravity and with re-orientation.

3 Molding in Three Dimensions

In this section, we summarize the three-dimensional aspects of the mold filling problem. These results are relevant to gravity-casting of 3-dimensional objects using molten metal for example.

3.1 Preliminaries

A *polyhedron* in E^3 is a solid whose surface consists of a number of polygonal faces. A polyhedron is *simple* if its surface can be deformed continuously into the surface of a sphere. A mold will be modelled by a simple polyhedron. As we are dealing only with simple polyhedra, we will refer to them as polyhedra.

A polyhedron partitions the space into two disjoint domains, the *interior* (bounded) and the *exterior* (unbounded). We will denote the open interior of the polyhedron P by $int(P)$, the boundary by $bd(P)$, and the open exterior by $ext(P)$. The boundary is considered part of the polyhedron; that is, $P= int(P) \cup bd(P)$.

If we intersect a polyhedron with an arbitrary plane, the result is a collection (possibly empty) of simple polygons (or line segments or points) lying on the plane. A polygon in this collection will be referred to as a *sectional* polygon. Notice that a sectional polygon divides the polyhedron into two simple polyhedra. In this sense a sectional polygon is the three dimensional equivalent to a chord in a polygon in 2 dimensions.

We assume the gravity casting model. When a direction of gravity is not specified, it is assumed that gravity points in the negative z-direction. Thus, if only one pin gate is used, we assume it to be a point on the boundary with the highest z-coordinate, since otherwise, the polyhedron cannot be completely filled. Unless stated otherwise, we will refer to *k-fillable* as filling from a fixed orientation.

3.2 The Decision Problem

Theorem 3.1 (Bose, van Kreveld and Toussaint [BVT93], Fekete and Mitchell [FM93]) A polyhedron P is 1-fillable if and only if the orientation of P yields precisely one local maximum in direction $+z$.

To test the 1-fillability of a polyhedron P with respect to gravity g, only the number of local maxima with respect to g need to be determined. Deciding if a vertex is a local maximum can be accomplished in time linear in the degree of the vertex. This immediately gives a linear time algorithm to determine whether or not a polyhedron is 1-fillable from a fixed orientation.

Theorem 3.2 (Bose, van Kreveld and Toussaint [BVT93]; Fekete and Mitchell [FM93]) Given a polyhedron P, in $O(n)$ time it can be determined whether or not the polyhedron is 1-fillable with respect to gravity.

3.3 Determining all Directions of Fillability

Suppose that we are given a polyhedron and asked whether there exists a direction of gravity allowing the polyhedron to be 1-fillable with respect to that direction. An $O(n^2)$ time algorithm exists to find the orientation of a given polyhedron P that minimizes the number of venting holes needed to ensure a complete fill from a fixed orientation. The algorithm has two stages. In the first stage, the fillability problem is transformed to a planar problem for a set of convex (possibly unbounded) polygons that cover the plane.

Lemma 3.1 (Bose, van Kreveld and Toussaint [BVT93]) In $O(n)$ time, one can transform the problem of k-fillability to the problem of finding a point in the plane covered by only k convex polygons.

Each convex polygon represents the set of directions that make the associated vertex a local maximum. Accordingly, the next step in the algorithm involves solving the following problem: 'Given a set Q of n convex, but not necessarily bounded, polygons in the plane, with total complexity $O(n)$, find a point that is covered by the minimum number of polygons of Q.' The cell in the subdivision induced by Q covered by the minimum number of polygons represents the set of directions that has the minimum number of local maxima. The time to find this cell is bounded by the total number of cells which is $O(n^2)$.

Theorem 3.3 (Bose, van Kreveld and Toussaint [BVT93]; Fekete and Mitchell [FM93]) Given a simple bounded polyhedron P in 3-space, one can find in $O(n^2)$ time an orientation for P such that P is fillable with the minimum number of venting holes.

3.4 Fillability of Certain Classes of Polyhedra

We outline the relationships between the notion of fillability and certain known classes of restricted polyhedra. These results are relevant to the manufacturing industry because in practice many objects are not modelled by polyhedra of arbitrary shape complexity.

A polyhedron P is *weakly monotonic* in direction l if there exists a direction l such that the intersection, of each plane orthogonal to l that intersects P, is a *simple polygon* (or a line segment or point). The direction l is referred to as the direction of monotonicity.

Note that there exist many different classes of simple polygons [Or87], [PS85], [To90], [Tou]. By substituting one of these classes for the word *simple* in the above definition, a score of families of *weakly monotonic* polyhedra are obtained. Thus we say that if all the intersections are *convex* polygons, we have a weakly monotonic polyhedron in the *convex sense*. If the intersections are *monotone* polygons, then we have a weakly monotonic polyhedron in the *monotone sense*, and so on.

Theorem 3.4 (Bose, van Kreveld and Toussaint [BVT93]) A weakly monotonic polyhedron P is 1-fillable if it is oriented such that gravity points in the direction of monotonicity.

A polyhedron P is *facet visible* if there is a facet of the polyhedron from which every point in the polyhedron is weakly visible. A polyhedron P is *open-facet visible* if there is a facet f in P such that every point p is visible from some point x on f that is not on the boundary of the facet.

Theorem 3.5 (Bose, van Kreveld and Toussaint [BVT93]) An open-facet visible polyhedron P is 1-fillable.

Theorem 3.6 (Bose, van Kreveld and Toussaint [BVT93]) Every polyhedron that is weakly visible from a sectional polygon is 2-fillable with re-orientation.

A *star-shaped polyhedron* is a polyhedron that contains at least one point x from which all points of the polyhedron are visible. Such a polyhedron may not necessarily be 1-fillable. In fact, if a star-shaped polyhedron is filled from one fixed orientation, it may need $\Omega(n)$ venting holes.

Theorem 3.7 (Bose, van Kreveld and Toussaint [BVT93]) A star-shaped polyhedron is not necessarily 1-fillable but can always be 2-filled with re-orientation in $O(n)$ time.

We can also classify polyhedra by their link diameter. A polyhedron P has *link diameter* k if every pair of points in P can be connected by a polygonal path with at most k edges that does not leave the polyhedron.

Theorem 3.8 (Fekete and Mitchell [FM93]) A polyhedron with link diameter 1 is 1-fillable.

Theorem 3.9 (Fekete and Mitchell [FM93]) A polyhedron with link diameter 2 is 2-fillable with re-orientation.

Theorem 3.10 (Fekete and Mitchell [FM93]) There exist polyhedra with link diameter 3 that require $\Omega(n)$ fillings, even if they are weakly externally visible.

4 Casting

Casting consists of filling the region bounded by two or more cast parts with a material such as a molten metal, after which the cast parts are removed. The removal of the cast parts without breaking them imposes certain restrictions on the shape of the object to be constructed. For *sand casting* (see e.g. [El88, WO81]), only two cast parts are used. To construct the cast parts, a prototype of the object is first obtained. The prototype is then divided into two parts along a plane. The face along which the part is cut, is referred to as the base. The first cast part is made by placing the base of the part on a flat surface, and then adding sand around it. The part is then rotated such that the base is facing up. The other part is placed such that the bases coincide and the second cast part is built by adding sand around this part. An opening into the cavity is maintained during the construction of the second cast part. This completes the construction of the cast of the prototype object. To build a metal rendition of the prototype object with this cast, liquid metal is poured into the opening until it fills the cavity. After the metal solidifies, the cast parts are removed from the object. To be able to re-use the cast parts when manufacturing many objects in this way, it is necessary that the two cast parts be removed without breaking the cast [El88], [WO81], [Is87], [Whe87]. Therefore for many different manufacturing methods involving casting, the geometry of the object determines its feasibility of construction.

We note that more complicated objects can be made by using cores and inserts [El88], [Is87], [WO81], [Whe87]. However, their use slows down the manufacturing process and makes it more costly. Therefore to be cost efficient, cores and inserts should be avoided.

4.1 Casting Two-dimensional Objects

A polygon is considered to be castable if the boundary can be decomposed into two chains such that each chain can be translated away from the polygon without intersecting the interior. Rosenbloom and Rappaport [RR92] proved the following.

Theorem 4.1 (Rosenbloom and Rappaport [RR92]) A simple polygon is castable if and only if its boundary can be decomposed into two monotone chains.

This characterization gives rise to a simple O(n) time algorithm to decide whether a simple polygon is castable.

4.2 Casting Three-dimensional Objects

For a polyhedron P, let ∂P denote the boundary of P, and for a non-vertical plane h, let h^+ and h^- refer to the open half spaces above and below h [BBV93]. To manufacture a polyhedron P that is castable, first a casting plane h for P is determined. Then the two cast parts are made from the prototype halves $h^+ \cap \partial P$ and $h^- \cap \partial P$. Since P is castable, the prototype halves can be removed from the cast parts, and later the manufactured object can be removed from the cast parts. Although the algorithms in [BBV93] deal with the sand casting process, they can be applied to related processes as well. In [BBV93] three versions of the castability problem are considered. They differ in the way the cast may be removed from the polyhedron P.

1. The two cast parts must be removed from P by one translation each, in opposite directions, and normal to the casting plane (orthogonal cast removal).
2. The two cast parts must be removed from P by one translation each, and in opposite directions (opposite cast removal).
3. The two cast parts must be removed from P by one translation each, in arbitrary directions (arbitrary cast removal).

In the manufacturing industry, developing machines that perform orthogonal and opposite cast removal is much simpler than producing machines that perform arbitrary cast removal. In fact, opposite cast removal seems to be the most popular technique used ([Pri87], [PC93]). Furthermore, if orthogonal or opposite cast removal is possible, it can be determined more efficiently. We summarize the complexity of the different algorithms [BBV93] developed for the casting problem.

	orthogonal	opposite	arbitrary
convex polyhedra	$O(n \log^2 n)$	$O(n \log^2 n)$	$O(n^2 \log n)$
simple polyhedra	$O(n^2)$	$O(n^2)$	$O(n^2 \log n)$

5 Stereolithography

In this section, we consider the problem of deciding whether or not a design is feasible for a CAD/CAM system developed and patented by 3D Systems of Sylmar, CA that employs a process called *stereolithography*. For details the reader is referred to [ABB93]. The components of the stereolithography manufacturing process consist of a vat of liquid photocurable plastic, a computer controlled table T on a stand S that can be moved up and down in the vat and a laser L above the vat that can shine on the surface of the liquid plastic and can move in a horizontal plane. The system works as follows. At the first step the table is just below the surface of the plastic and the laser is controlled to move about so that the light shines on the surface of the plastic and *draws* the bottom-most cross-section of the object A being built. When the laser light contacts the plastic, the plastic solidifies and so the first cross-section of the object is formed and rests on the table. At the next step the table is lowered a small amount to allow liquid to cover the hardened layer and the laser then draws the next cross-section of the object. The light from the laser penetrates the liquid just deep enough so that this cross-section is welded to the lower cross-section produced at the previous step. This process is repeated until the entire object is formed. The direction given by a normal to the table pointing to the laser is called the *direction of formation* for the object. There are some objects that can be formed only if the direction of formation is chosen correctly. Naturally, there are some objects that can not be formed using stereolithography regardless of the direction of formation chosen, such as a sphere.

Theorem 5.1 ([ABB93]) In $O(n)$ time the feasibility of a polyhedral object with n vertices can be determined and a valid base identified when the object is feasible.

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