Proximity Graphs on Line Segments

Prosenjit Bose Godfried Toussaint

School of Computer Science
McGill University
3480 University Street
Montréal, Québec, Canada
H3A 2A7

Many problems in pattern recognition and classification, geographic variation analysis, geographic information systems, computational geometry, computational morphology, and computer vision model the underlying *structure* (also referred to as the *skeleton* or *internal shape*) of a set of data points by means of a *proximity graph*. In a proximity graph, two points are joined by an edge if they are deemed "close" by some proximity measure. It is the measure that determines the type of graph that results. Many different measures of proximity have been defined. For example, the closeness of two points can be measured by looking at the smallest closed disk that contains them; if the disk does not contain any other point of the set, the two points are connected by an edge. The resulting graph is known as the *Gabriel Graph*. For a comprehensive survey of proximity graphs on point sets, the reader is referred to Jaromczyk and Toussaint [JT92].

The study of proximity graphs defined on sets of disjoint line segments has received considerably less attention. Although for point sets the definition of proximity measures seems quite straightforward, this is no longer the case with line segments. For example, it seems natural that a proximity graph defined on a set of line segments have its closest pair of vertices as an edge. Consider the situation depicted in Figure 1. The closest pair of vertices is formed by vertices b and c. However, the line segment $[ef]$ presents a barrier. Therefore, in this case, it may be better to omit the closest pair.

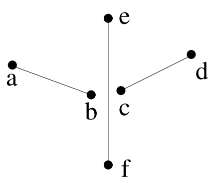

Figure 1: Closest pairs may not be part of the proximity graph.

Lee [Lee78] first addressed these problems in the context of Delaunay triangulations. His goal was the following: Given a set of n vertices in the plane together with a set of non-crossing edges, find a triangulation that includes the prespecified edges and is as close as possible to the Delaunay triangulation. Lee defined the following as a *constrained Delaunay triangulation* (CDT).

Definition 1 *Let G be a straight-line planar graph. A triangulation T is a constrained Delaunay triangulation of G if each edge of G is an edge of T and for each remaining edge e of T there exists a circle C with the following properties:*

1. *the endpoints of edge e are on the boundary of C, and*
2. *if any vertex v of G is in the interior of C then it cannon be "seen" from at least one of the endpoints of e (i.e. if line segments are drawn from v to each endpoint of e then at least one of the line segments crosses an edge of G).*

Note that if G has no edges, then the constrained Delaunay triangulation is the same as the (unconstrained) Delaunay triangulation (see Figure 2).

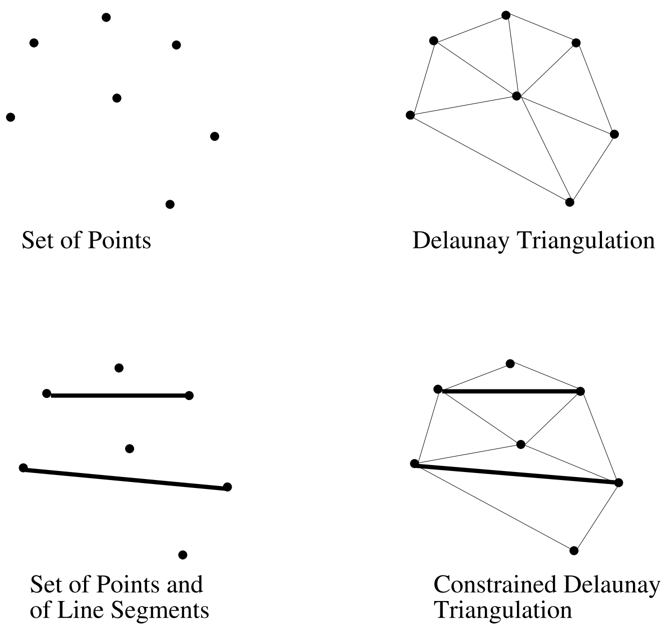

Figure 2: Delaunay triangulation and constrained Delaunay triangulation.

Lee's algorithm to compute the constrained Delaunay triangulation runs in $O(n^2 \log n)$ time. Independently, Chew [Che87, Che89], and Wang and Schubert [WS87] presented optimal $O(n \log n)$ time algorithms. Chew used a divide-and-conquer approach to compute the constrained Delaunay triangulation whereas Wang and Schubert took advantage of the duality between Voronoi diagrams and Delaunay triangulations. For the special case where the line segments form a simple polygon, Schachter [Sch78] gives an $O(nr)$ time algorithm to compute the constrained Delaunay triangulation where n is the number of line segments and r is the number of reflex vertices of the simple polygon. Several other algorithms, variations and extensions have been studied since [Ber88, CR90, GR91, Jun88, KM92, MV93, Sei88, Slo91, Wan93, WT92].

Both Jennings and Lingas [JL92] and Bose and Toussaint [BT92] independently showed that the Euclidean minimum spanning tree of a set of disjoint line segments is crossing free. In fact, Chew [Che87, Che89] pointed out that the constrained Euclidean minimum spanning tree is a subgraph of the constrained Delaunay triangulation. A natural extension to these ideas is to consider the relative neighborhood graph and the Gabriel graph. In this context, ElGindy and Toussaint [ET88] first addressed the problem of computing the relative neighborhood graph of a set of line segments where the line segments form a simple polygon. They presented an $O(n^2)$ time algorithm to compute this graph. Subsequently, Su and Chang [SC91] presented optimal $O(n \log n)$ time algorithms to compute both the constrained relative neighborhood graph and the constrained Gabriel graph by essentially pruning the constrained Delaunay triangulation.

The main emphasis in the study of proximity graphs constrained to a set of line segments has been algorithmic. There has been some work done in characterizing proximity graphs of point sets (see [LS93, BBL93, BBL94, GDBL94, FHM93, MJMar, MQar]), but almost no work has been done from a graph-theoretic point of view for proximity graphs of line segments. The paper of Bose and Toussaint [BT92], who showed that the maximum degree of the Euclidean minimum spanning tree of a set of disjoint line segments is at most seven, seems to be the only work that has addressed these problems to date. There are many open problems in this area. We conclude with a few of them.

1. What is the maximum degree of a vertex of the relative neighborhood graph, Gabriel graph and Delaunay triangulation of a set of disjoint line segments?

2. Which trees can be represented as the Euclidean minimum spanning tree of a set of disjoint line segments?

3. Which triangulations can be represented as the Delaunay triangulation of a set of disjoint line segments?

4. Which graphs can be represented as the Gabriel graph or relative neighborhood graph of a set of disjoint line segments?

References

[Ber88] J. Bernal. Constructing Delaunay triangulations for sets constrained by line segments. Technical Report NIST/TN-1252, Mathematical Analysis Div., National Inst. of Standards and Technology (NEL), Gaithersburg, MD, September 1988.

[BBL93] P. Bose W. Lenhart and G. Liotta. Characterizing proximity trees. *Algorithmica*, to appear. Also available as: Technical Report no. SOCS-93.9, School of Computer Science, McGill Unibersity, December 1993.

[BBL94] P. Bose, G. Di Battista W. Lenhart and G. Liotta. Proximity constraints and representable trees. To appear in *Proc. Graph Drawing '94*, Princeton, New Jersey, 1994.

[BT92] P. Bose and G. Toussaint. Growing a tree from its branches. In *Proc. First Pacific Conf. on Computer Grapics and Appl.*, pages 91–103, 1993 Also available as: McGill Technical Report no. SOCS 92.12, McGill University, 1992.

[Che87] L. P. Chew. Constrained Delaunay triangulations. In *Proc. 3rd Annu. ACM Sympos. Comput. Geom.*, pages 215–222, 1987.

[Che89] L. P. Chew. Constrained Delaunay triangulations. *Algorithmica*, 4:97–108, 1989.

[CR90] A. K. Cline and R. J. Renka. A constrained 2-dimensional triangulation and the solution of closest node problems in the presence of barriers. *SIAM J. Numer. Anal.*, 27(5):1305–1321, 1990.

[ET88] H. A. ElGindy and G. T. Toussaint. Computing the relative neighbor decomposition of a simple polyg on. In G. T. Toussaint, editor, *Computational Morphology*, pages 53–70. North-Holland, Amsterdam, Netherlands, 1988.

[FHM93] M.J. Lipman F. Harary, M.S. Jacobson and F.R. McMorris. Abstract sphere-of-influence graphs. *Math. Comput. Modelling*, 17:77–83, 1993.

[GDBL94] W. Lenhart G. Di Battista and G. Liotta. Proximity drawability: A survey. To appear in *Proc. Graph Drawing '94*, Princeton, New Jersey, 1994.

[GR91] T. Gonzalez and M. Razzazi. Properties and algorithms for constrained Delaunay triangulations. In *Proc. 3rd Canad. Conf. Comput. Geom.*, pages 114–117, 1991.

[JL92] E. Jennings and A. Lingas. On the relationship among constrained geometric structures. In *Proc. 3rd Annu. Internat. Sympos. Algorithms Comput. (ISAAC '92)*, pages 289–298, 1992.

[JT92] J. W. Jaromczyk and G. T. Toussaint. Relative neighborhood graphs and their relatives. *Proc. IEEE*, 80(9):1502–1517, September 1992.

[Jun88] Dz-Mou Jung. An optimal algorithm for constrained Delaunay triangulation. In *Proc. 26th Allerton Conf. Commun. Control Comput.*, pages 85–86, Urbana, IL, 1988. Univ. Illinois.

[KM92] T. C. Kao and D. M. Mount. Incremental construction and dynamic maintenance of constrained Delaunay triangulations. In *Proc. 4th Canad. Conf. Comput. Geom.*, pages 170–175, 1992.

[Lee78] D.T. Lee. Proximity and reachibility in the plane. Technical Report R-831, Coordinated Science Laboratory, University of Illinois, 1978.

[LS93] A. Lubiw and N. Sleumer. Maximal outerplanar graphs are relative neighborhood graphs. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 198–203, Waterloo, Canada, 1993.

[MJMar] M.J. Lipman M.S. Jacobson and F.R. McMorris. Trees that are sphere-of-influence graphs. *Appl. Math. Letters*, to appear.

[MQar] T.S. Michael and T. Quint. Sphere-of-influence graphs: a survey. *Congr. Numer.*, to appear.

[MV93] J.-M. Moreau and P. Volino. Constrained Delaunay triangulation revisited. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 340–345, Waterloo, Canada, 1993.

[SC91] Tung-Hsin Su and Ruei-Chuan Chang. Computing the constrained relative neighborhood graphs and constrained gabriel graphs in Euclidean plane. *Pattern Recogn. Lett.*, 24(3):221–230, 1991.

[Sch78] B.J. Schachter. Decomposition of polygons into convex sets. *IEEE Trans. Comput.*, 27(11):1078–1082, November 1978.

[Sei88] R. Seidel. Constrained Delaunay triangulations and Voronoi diagrams. Report 260, IIG-TU, Graz, Austria, 1988.

[Slo91] S. W. Sloan. A fast algorithm for generating constrained Delaunay triangulations. Research Report 065.07.1991, The University of Newcastle Department of Civil Engineering and Surveying, New South Wales, 1991.

[Wan93] C. A. Wang. Efficiently updating the constrained delaunay triangulations. *BIT*, 33:238–252, 1993.

[WS87] C. A. Wang and L. Schubert. An optimal algorithm for constructing the Delaunay triangulation of a set of line segments. In *Proc. 3rd Annu. ACM Sympos. Comput. Geom.*, pages 223–232, 1987.

[WT92] C. A. Wang and Y. H. Tsin. Efficiently updating constrained Delaunay triangulations. In *Proc. 4th Canad. Conf. Comput. Geom.*, pages 176–181, 1992.